Wild solenoids

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Cantor laminations

Let M be a compact connected metrizable topological space with a foliation $\mathcal{F}$.

A Cantor lamination

The space $(M, \mathcal{F})$ is a Cantor lamination, M admits a foliated atlas $\varphi_i : U_i \to (-1,1)^n \times Z_i$, where Z_i is a Cantor set, $1 \leq i \leq \nu$.

Note: The leaves of $\mathcal{F}$ are path-connected components of M.

Associated to (φ_i, U_i) there is the *holonomy pseudogroup*

$$\Gamma = \{h_{i_k i_{k-1}} \circ \cdots \circ h_{i_1 i_0} \mid k \geq 1, U_{i_j} \cap U_{i_{j-1}} \neq \emptyset\},$$

where h_{ij} are the restrictions of the transition maps to $\varphi_j(U_i \cap U_j) \subset Z_j$.

Invariants of Cantor laminations

In a foliated space $(M, \mathcal{F})$, the holonomy pseudogroup Γ depends on the choice of the transverse sections $Z = \bigcup_i Z_i$.

Motivating problem

Given a Cantor lamination $(M, \mathcal{F})$ with holonomy pseudogroup (Z, Γ), find (algebraic) invariants of $\mathcal{F}$, which do not depend on the choice of Z.

In the rest of the talk, we restrict to the case when (Z, Γ) is *equicontinuous*, that is, Z admits a metric d, so that for any $\epsilon > 0$ there is $\delta > 0$ such that for all $g, h \in \Gamma$ and all $x, y \in Z$ with $d(x, y) < \delta$ we have $d(g(x), g(y)) < \epsilon$.

Examples of Cantor laminations with equicontinuous dynamics are *solenoids*.

Example in dimension 1: the Vietoris solenoid

Let $f^i_{i-1} : \mathbb{S}^1 \to \mathbb{S}^1$ be a p_i-to-1 self-covering map of a circle of length 1.

Let $\mathbf{P} = (p_1, p_2, \ldots)$ be an infinite sequence, $p_i > 1$. A Vietoris solenoid is the inverse limit space

$$\Sigma_{\mathbf{P}} = \{(x_i) = (x_0, x_1, x_2, \ldots) \mid f^i_{i-1}(x_i) = x_{i-1} \} \subset \prod_{i \geq 0} \mathbb{S}^1$$

with subspace topology from the Tychonoff topology on $\prod_{i \geq 0} \mathbb{S}^1$.

Let $s \in \mathbb{S}^1$, then the fibre $F_0 = \{(s, x_1, x_2, \ldots)\}$ is a Cantor section, transverse to the foliation by path-connected components.

The fundamental group $\pi_1(\mathbb{S}^1, 0) = \mathbb{Z}$ acts on F_0 via lifts of paths in $\mathbb{S}^1$, so the pseudogroup action (F_0, Γ) is induced by a group action $(F_0, \mathbb{Z})$.

Weak solenoids

Let M_0 be a closed manifold, $x_0 \in M_0$ be a point, and let $G = \pi_1(M_0, x_0)$ be the fundamental group.

Let $f_i^{i+1} : M_{i+1} \to M_i$ be a finite-to-one covering, $f_i^{i+1}(x_{i+1}) = x_i$.

The inverse limit space, called a *weak solenoid*, and defined by

$$\mathcal{S}_{\mathcal{P}} = \varprojlim \{f_i^{i+1} : M_{i+1} \to M_i\} = \{(x_0, x_1, x_2, \cdots) \mid f_i^{i+1}(x_{i+1}) = x_i\}$$

is a fibre bundle with Cantor fibre F_0.

The fundamental group G of M_0 acts on F_0 by path-lifts.

The pseudogroup action (F_0, Γ) is induced by the group action (F_0, G).

Solenoids in dimension 2 and higher I

Weak solenoids were introduced by **McCord 1965**, and studied in
- **Schori 1966**,
- **Rogers and Tollefson 1970, 1971, 1971/72**.

Recall that a topological space M is *homogeneous* if for every $x, y \in X$ there exists a homeomorphism $\phi : M \to M$ such that $\phi(x) = y$.

If M is a Cantor lamination, every homeomorphism $\phi : M \to M$ is a foliated map, as it preserves path-connected components.

McCord 1965 found sufficient conditions for a solenoid M to be a homogeneous foliated space.

Schori 1966, **Rogers and Tollefson 1970, 1971, 1971/72** found examples of non-homogeneous solenoids, and studied maps between solenoids.

Solenoids in dimension 2 and higher II

Fokkink and Oversteegen 2002 found an algebraic criterion for a weak solenoid to be a homogeneous space, and constructed the first example of a weak solenoid which has simply connected path-connected components but is not a homogeneous space.

Dyer, Hurder and Lukina 2016(1), 2017, 2016(2) explored an algebraic invariant of non-homogeneous solenoids, called the *discriminant group*.

Today, I will talk about a generalization of this invariant, called the *asymptotic discriminant*, and how it classifies non-homogeneous solenoids. The talk is based on the recent joint work with Steven Hurder **arXiv:1702.03032**.

Group chains: a tool

Let $\mathcal{S}_{\mathcal{P}} = \varprojlim \{f_i^{i+1} : M_{i+1} \to M_i\}$ be a solenoid, $G_0 = \pi_1(M_0, x_0)$ and

$$G_{i+1} = (f_0^{i+1})_* \pi_1(M_{i+1}, x_{i+1}) \subset G_0 \ .$$

Then $G = G_0 \supset G_1 \supset G_2 \supset \cdots$ is a chain of subgroups of finite index.

The quotient G/G_i is a finite set, the inclusions $G_{i+1} \subset G_i$ induce surjective maps $G/G_{i+1} \to G/G_i$, and there is a homeomorphism

$$\phi : F_0 \to G_\infty = \varprojlim \{G/G_i \to G/G_{i-1}\}$$

equivariant with respect to (F_0, G) and the action

$$G \times G_\infty \to G_\infty : (h, (eG_0, g_1 G_1, g_2 G_2, \ldots)) \mapsto (hG_0, hg_1 G_1, hg_2 G_2, \ldots)$$

Thus (G_∞, G) models the action (F_0, G, Φ).

Profinite group acting on F_0

Let $C_i = \bigcap_{g \in G} g G_i g^{-1}$, then C_i is a finite index normal subgroup of G,

$$\text{then } C_\infty = \varprojlim\{G/C_i \to G/C_{i-1}\} \text{ is a profinite group,}$$

acting transitively on $G_\infty \cong F_0$, with the isotropy group

$$\mathcal{D}_x = \varprojlim\{G_i/C_i \to G_{i-1}/C_{i-1}\}$$

Theorem (Dyer, Hurder, L. 2016(1))

The profinite group C_∞ is isomorphic to the *Ellis group* of the action (F_0, G, Φ), and $\mathcal{D}_x$ is isomorphic to the isotropy group of the Ellis group action on F_0.

We call $\mathcal{D}_x$ the *discriminant group* of the action.

Since $\mathcal{D}_x$ is a profinite group, it is either finite, or a Cantor group.

Algebra versus topology

Let $\mathcal{P} = \{f_i^{i+1} : M_{i+1} \to M_i \mid i \geq 0\}$ be a presentation for $\mathcal{S}_{\mathcal{P}}$.

The truncated presentation $\mathcal{P}_n = \{f_i^{i+1} : M_{i+1} \to M_i \mid i \geq n\}$ determines a solenoid $\mathcal{S}_{\mathcal{P}_n}$, homeomorphic to $\mathcal{S}_{\mathcal{P}}$.

The truncated presentation corresponds to the truncated group chain $\{G_i\}_{i \geq n}$ with discriminant group $\mathcal{D}_n$.

Main question of the talk

What is the relationship between the discriminant groups $\mathcal{D}_n$, for $n \geq 0$?

Example: If a solenoid is homogeneous (for example, a Vietoris solenoid), then $\mathcal{D}_n$ is trivial for all $n \geq 1$.

In general, this need not be the case.

Why the discriminant group changes

Let $\{G_i\}_{i\geq 0}$ be a group chain, and $\mathcal{D}_x = \varprojlim\{G_{i+1}/C_{i+1} \to G_i/C_i\}$ be the discriminant group. Recall that $C_i = \bigcap_{g \in G} g_i G_i g_i^{-1}$.

Let $\{G_i\}_{i\geq n}$ be a truncated group chain for $n \geq 0$. Then it's maximal normal subgroup in G_n is

$$E_{n,i} = \bigcap_{g \in G_n} g_i G_i g_i^{-1}, \text{ and } \mathcal{D}_n = \varprojlim\{G_{i+1}/E_{n,i+1} \to G_i/E_{n,i}\}$$

Thus $C_i \subset E_{n,i} \subset E_{n+k,i}$, also $\mathcal{D}_n = \varprojlim\{G_{i+1}/E_{n,i+1} \to G_i/E_{n,i}\}$, and there are a surjective maps

$$\Lambda_{n+k,n} : \mathcal{D}_n \to \mathcal{D}_{n+k}.$$

Stable and wild group actions

Definition: Stable group actions (Dyer, Hurder, L. 2016(2))

Let (X, G, Φ) be a group action with group chain $\{G_i\}_{i \geq 0}$. Then (X, G, Φ) is *stable* if there exists $n \geq 0$ such that for all $m \geq n$ the restriction

$$\Lambda_{m,n} : \mathcal{D}_n \to \mathcal{D}_m$$

is an isomorphism. Otherwise, the action is said to be *wild*.

Example: Vietoris solenoids are stable, with trivial discriminant group.

Realization theorems for stable actions

Every finite group can be realized as a discriminant group of a stable equicontinuous action on a Cantor set.

Theorem (Dyer, Hurder, L. 2016(2))

Given a finite group F, there exists a group chain $\{G_i\}_{i \geq 0}$ in $SL(n, \mathbb{Z})$ for n large enough, such that for any truncated chain $\{G_i\}_{i \geq n}$ the discriminant group $\mathcal{D}_n \cong F$.

Every separable profinite group can be realized as a discriminant group of a stable equicontinuous action on a Cantor set.

Theorem (Dyer, Hurder, L. 2016(2))

Given a separable profinite group K, there exists a finitely generated group G and a group chain $\{G_i\}_{i \geq 0} \subset G$, such that for any truncated chain $\{G_i\}_{i \geq n}$ the discriminant group $\mathcal{D}_n \cong K$, and $\mathcal{D}_n \to \mathcal{D}_{n+1}$ is an isomorphism.

Tail equivalence of discriminant groups

Let $\mathcal{A} = \{\phi_i : A_i \to A_{i+1}\}$ and $\mathcal{B} = \{\psi_i : B_i \to B_{i+1}\}$ be sequences of surjective group homomorphisms. Then $\mathcal{A}$ and $\mathcal{B}$ are tail equivalent if there are infinite subsequences A_{i_n} and B_{j_n} such that the sequence

$$A_{i_0} \to B_{j_0} \to A_{i_1} \to B_{j_1} \to \cdots$$

is a sequence of surjective group homomorphisms.

Asymptotic discriminant (Hurder and L. 2017)

Let $\{G_i\}_{i \geq 0}$ be a group chain with discriminant group $\mathcal{D}_0$, and let $\mathcal{D}_n$ be the discriminant group of the truncated chain $\{G_i\}_{i \geq n}$. The *asymptotic discriminant* of the group chain $\{G_i\}_{i \geq 0}$ is the tail equivalence class of the sequence of surjective group homomorphisms $\{\mathcal{D}_n \to \mathcal{D}_{n+1} \mid n \geq 0\}$.

These sequences are distinct from group chains $\{G_i\}_{i \geq 0}$: the inclusion $G_{i+1} \to G_i$ is not surjective.

Stability of the asymptotic discriminant

A sequence $\mathcal{A} = \{\phi_i : A_i \to A_{i+1}\}$ is *constant* if ϕ_i is a group isomorphism for $i \geq 0$.

Asymptotically constant sequence

A sequence $\mathcal{B} = \{\phi_i : B_i \to B_{i+1}\}$ is *asymptotically constant* if it is tail equivalent to a constant sequence.

Now the stability of the action (X, G, Φ) can be defined in terms of the asymptotic discriminant.

Lemma

An equicontinuous group action (X, G, Φ) with associated group chain $\{G_i\}_{i \geq 0}$ is stable if and only if it's asymptotic discriminant $\mathcal{D} = \{\mathcal{D}_n \to \mathcal{D}_{n+1}\}$ is tail equivalent to a constant sequence.

Wild solenoids

Question

Does the asymptotic discriminant really distinguish between different group actions?

The answer is yes.

Theorem (Hurder and L. 2017)

For $n \geq 3$, let $G \subset \mathbf{SL}(n, \mathbb{Z})$ be a torsion-free subgroup of finite index. Then there exists uncountably many distinct homeomorphism types of weak solenoids which are wild, all with the same base manifold M_0 with fundamental group G.

In the remaining part of the talk, I will construct a family of examples which give this theorem.

Lenstra's construction

Let G be a finitely generated and residually finite group, and $\widehat{G}$ be its profinite completion. Then G embeds as a dense subgroup of $\widehat{G}$.

Let D be a closed subgroup of $\widehat{G}$ such that $\bigcap_{g \in G} gDg^{-1} = \{e\}$.

Let $\{U_i\}$ be a clopen neighborhood system of the identity in $\widehat{G}$, where U_i is normal and of finite index, and $\bigcap_{i \geq 0} U_i = \{e\}$.

$W_i = DU_i$ is a clopen subgroup of $\widehat{G}$, such that $\bigcap_{i \geq 0} W_i = D$. Set

$$G_i = W_i \cap G.$$

Proposition

The group chain $\{G_i\}_{i \geq 0}$, constructed as above, has the discriminant group $\mathcal{D}_x \cong D$.

Wild solenoids I

Consider congruence subgroups of $\mathbf{SL}_N(\mathbb{Z})$, $N \geq 3$,

$$\Gamma_N(M) = Ker\{\mathbf{SL}_N(\mathbb{Z}) \to \mathbf{SL}_N(\mathbb{Z}/M\mathbb{Z})\}.$$

For $M \geq 3$, $\Gamma_N(M)$ is torsion-free.

Let G be a finite-index torsion-free subgroup of $\mathbf{SL}_N(\mathbb{Z})$.

By the congruence subgroup property, $G \supset \Gamma_N(M)$ for some $M \geq 3$.

Let $\widehat{H} = \prod_{p_i \geq M} \mathbf{SL}_N(\mathbb{Z}/p_i\mathbb{Z})$, p_i are distinct primes.

Lemma

Let $Q : G \to \widehat{H}$ be the diagonal embedding. Then $Q(G)$ is dense in $\widehat{H}$.

Wild solenoids II

For $p_i \geq M$, let A_{p_i} be a non-trivial subgroup of $\mathbf{SL}_N(\mathbb{Z}/p_i\mathbb{Z})$ such that

$$\bigcap_{g \in \mathbf{SL}_N(\mathbb{Z}/p_i\mathbb{Z})} g A_{p_i} g^{-1} = \{e\}$$

(for example, A_{p_i} may be in $\mathbf{Alt}_N$ embedded into $\mathbf{SL}_N(\mathbb{Z}/p_i\mathbb{Z})$).

Let $D = \prod_{p_i \geq M} A_{p_i}$, and set $G_\ell = G \cap U_\ell$, where

$$U_\ell = \prod_{m \leq i < \ell} \{e_{p_i}\} \times \prod_{i \geq \ell} \mathbf{SL}_N(\mathbb{Z}/p_i\mathbb{Z}).$$

Proposition

The action (G_∞, G) with group chain $\{G_\ell\}_{\ell \geq 1}$ defined as above has the discriminant group $\mathcal{D}_0 = D$.

Wild solenoids III

For $n \geq 1$, consider truncated chains $\{G_\ell\}_{\ell \geq n}$. One can show that the discriminant of the action $(G_{n,\infty}, G_n)$ is

$$\mathcal{D}_n = \prod_{p_i \geq n} A_{p_i},$$

and so the asymptotic discriminant of the action (G_∞, G) is the tail equivalence class of the system $\{\mathcal{D}_n \to \mathcal{D}_{n+1}\}$.

Remark

For each $n \geq 1$, the kernel of $\mathcal{D}_n \to \mathcal{D}_{n+1}$ is non-trivial and equal to A_n. Thus the action (G_∞, G) is wild.

Wild solenoids IV

An uncountable number of distinct asymptotic discriminants is obtained by distinct choices of $\prod A_{p_i}$, for $N = 3$. For a prime p_i, let

$$A^1_{p_i} = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & a \\ 0 & 0 & 1 \end{bmatrix} \Bigg| \, a \in \mathbb{Z}/p_i\mathbb{Z} \right\},$$

$$A^2_{p_i} = \left\{ \begin{bmatrix} 1 & 0 & b \\ 0 & 1 & a \\ 0 & 0 & 1 \end{bmatrix} \Bigg| \, a, b \in \mathbb{Z}/p_i\mathbb{Z} \right\}.$$

Then $A^1_{p_i}$ has order p_i, and $A^2_{p_i}$ has order p_i^2.

Wild solenoids V

Let $\Sigma = \prod_{i \geq M} \{1, 2\}$, so $\mathcal{I} = (s_M, s_{M+1}, \ldots)$, and let

$$\mathcal{D}^{\mathcal{I}} = \prod_{i \geq M} A_{p_i}^{s_i}.$$

Proposition

Let $\mathcal{I}_1, \mathcal{I}_2 \in \Sigma$. Then the sequences of surjective homomorphisms $\{\mathcal{D}_n^{\mathcal{I}_1} \to \mathcal{D}_{n+1}^{\mathcal{I}_1}\}$ and $\{\mathcal{D}_n^{\mathcal{I}_2} \to \mathcal{D}_{n+1}^{\mathcal{I}_2}\}$ are tail equivalent if and only if there is $n > M$ such that for all $i \geq n$ we have $s_i = t_i$.

This shows that there is uncountably many distinct tail equivalence classes of wild solenoids.

Work in progress

1. **Conjecture:** Let $\mathcal{S}_{\mathcal{P}}$ be a weak solenoid with group chain $\{G_i\}_{i \geq 0}$, and suppose there exists a foliated embedding $\mathcal{S}_{\mathcal{P}} \to M$, where M is a smooth finite-dimensional manifold with a C^2-foliation $\mathcal{F}$. Then the asymptotic discriminant class of $\{G_i\}_{i \geq 0}$ is constant, that is, the group action on the fibre of the solenoid $\mathcal{S}_{\mathcal{P}}$ is stable.
2. In all examples of wild solenoids, constructed so far, the embedding $G \to Homeo(F_0)$, where G is the fundamental group of the base manifold, has an infinitely generated kernel. Find an example of a wild solenoid where the kernel is finitely generated, or show that this is not possible.

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