Obstructions to lifting cocycles on groupoids and the associated C^*-algebras

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Cocycles

- Let Γ be a locally compact Hausdorff groupoid and let G be a locally compact abelian group.
- Let $Z_\Gamma(G)$ be the collection of continuous cocycles $\phi : \Gamma \to G$, that is

$$\phi(\gamma_1\gamma_2) = \phi(\gamma_1) + \phi(\gamma_2).$$

- $Z_\Gamma(G)$ is an abelian group and the map $G \mapsto Z_\Gamma(G)$ is a functor.

Twists

Definition

Let A be an abelian group and Γ a groupoid. A *twist* by A over Γ is a central groupoid extension

$$\Gamma^0 \times A \xrightarrow{j} \Sigma \xrightarrow{\pi} \Gamma,$$

where $\Sigma^0 = \Gamma^0$, j is injective, π is surjective, and $j(r(\sigma), a)\sigma = \sigma j(s(\sigma), a)$.

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Fact

A twist Σ by A is properly isomorphic to a trivial twist if and only if there is a groupoid homomorphism $\tau : \Gamma \to \Sigma$ such that $\pi\tau = id_\Gamma$.

The group of twists

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Fact

If $f \in \mathrm{Hom}_\Gamma(A, B)$ and Σ is a twist by A, then there is a unique twist f_*B by B and a twist morphism $f_* : \Sigma \to f_*\Sigma$ which is compatible with f.

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Fact

$T_\Gamma(A)$ is an abelian group under the operation

$$[\Sigma] + [\Sigma'] = \nabla_*^A[\Sigma *_\Gamma \Sigma'],$$

where $\nabla^A : A \oplus A \to A$ is $\nabla^A(a, a') = a + a'$. Moreover, $A \mapsto T_\Gamma(A)$ is a half-exact functor.

Obstruction to lifting a cocycle

Example

- Let $p : B \to C$ be a homomorphism of abelian groups.
- Let $\phi \in Z_\Gamma(C)$.
- Define the *obstruction twist*

$$\Sigma_\phi = \{(\gamma, b) \in \Gamma \times B \,:\, \phi(\gamma) = p(b)\}.$$

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Lemma

1. Σ_ϕ *is a twist of* Γ *by* $A := \ker p$.
2. Σ_ϕ *is trivial if and only* ϕ *lifts to a* B*-valued cocycle.*
3. *If* Γ *is étale and* A *is discrete then* Σ_ϕ *is étale. Moreover, if* p *is surjective then the projection* $\pi_1 : \Sigma_\phi \to \Gamma$ *is a surjective local homeomorphism.*

Exact sequences

Lemma

If

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$$

is a short exact sequence of abelian groups, then there is an exact sequence

$$0 \to Z_\Gamma(A) \xrightarrow{i_*} Z_\Gamma(B) \xrightarrow{p_*} Z_\Gamma(C) \xrightarrow{\delta} T_\Gamma(A),$$

where $\delta(\phi) := [\Sigma_\phi]$ *for* $\phi \in Z_\Gamma(C)$

Example I

Example

- Let $A = \mathbb{Z}$, $B = \mathbb{R}$, and $C = \mathbb{T}$. Let X be a Hausdorff second countable locally compact space.
- Let $\mathcal{U} := \{U_i\}_{i \in I}$ be a locally finite cover of X. Set $U_{ij} := U_i \cap U_j$ for $i, j \in I$.
- Let $\Gamma_{\mathcal{U}} := \{(x, i, j) : x \in U_{ij}\}$.
- Suppose that $\lambda = \{\lambda_{ij}\}_{i,j \in I}$ is a Čech 1-cocycle with coefficients in $\mathcal{T}$.
- The map $\phi : \Gamma_{\mathcal{U}} \to \mathbb{T}$ given by $\phi((x, i, j)) = \lambda_{ij}(x)$ is a continuous groupoid 1-cocycle.

Example II

Example

- Suppose that each λ_{ij} lifts, that is, for each $i, j \in I$ there is a continuous function $\tilde{\lambda}_{ij} : U_{ij} \to \mathbb{R}$ such that $\lambda_{ij} = e \circ \tilde{\lambda}_{ij}$.
- The formula

$$(\lambda^\star)_{ijk}(x) := \tilde{\lambda}_{ij}(x) + \tilde{\lambda}_{jk}(x) - \tilde{\lambda}_{ik}(x)$$

  defines a Čech 2-cocycle with values in $\mathbb{Z}$
- One can defina a groupoid 2-cocycle $\phi^\star$ by the formula

$$\phi^\star((x, i, j), (x, j, k)) = (\lambda^\star)_{ijk}(x)$$

Example III

Example

- Let Σ be the twist by $\mathbb{Z}$ over $\Gamma_{\mathcal{U}}$ determined by $\phi^{\star}$:
  $\Sigma := \{(n,(x,i,j)) : n \in \mathbb{Z}, x \in U_{ij}\}$. The structure maps are given by

$$\begin{aligned} (m,(x,i,j))(n,(x,j,k)) &:= (m+n+(\lambda^{\star})_{ijk}(x),(x,i,k)) \\ (n,(x,i,j))^{-1} &:= (-n-(\lambda^{\star})_{jij}(x),(x,j,i)). \end{aligned}$$

- Then $\Sigma_{\phi} \simeq \Sigma$.

Strongly continuous action induced by a 1-cocycle

Lemma

Assume that C is a locally compact abelian group. Given $\phi \in Z_\Gamma(C)$, the map

$$\alpha_t^\phi(f)(\gamma) = \langle t, \phi(\gamma) \rangle f(\gamma),$$

for $f \in C_c(\Gamma)$, $t \in \widehat{C}$, and $\gamma \in \Gamma$ defines a strongly continuous action $\alpha^\phi : \widehat{C} \to \mathrm{Aut}\, C^*(\Gamma)$.

Main Theorem

Theorem

Let Γ *be a locally compact Hausdorff amenable groupoid endowed with Haar system* $\{\lambda^u\}_{u\in\Gamma^0}$ *and let*

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0,$$

be a short exact sequence of locally compact abelian groups. Let $\phi \in Z_\Gamma(C)$ *and let* Σ_ϕ *be the obstruction twist endowed with the natural Haar system. Then* $C^*(\Sigma_\phi)$ *is isomorphic to* $\mathrm{ind}_{\hat{C}}^{\hat{B}}\, C^*(\Gamma)$*.*

The induced algebra

Definition

If H is a closed subgroup of a locally compact group G and (D, H, α) is a dynamical system, the *induced algebra* is

$$\mathrm{ind}_H^G(D, \alpha) = \{f \in C^b(G, D) \,:\, f(sh) = \alpha_h^{-1}(f(s)) \\ \text{and } sH \mapsto \|f(s)\| \in C_0(G/H)\}.$$

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Fact

There is a canonical translation $\beta : G \to \operatorname{Aut}(\operatorname{ind}_H^G(D, \alpha))$ *given by* $\beta_g(f)(x) = f(x - g)$. *Moreover,* $D \rtimes_\alpha H$ *is strongly Morita equivalent to* $\operatorname{ind}_H^G(D, \alpha) \rtimes_\beta G$.

The induced algebra

- Point evaluation at 0 yields a surjective homomorphism $\pi : \mathrm{ind}_H^G(D, \alpha) \to D$ given by $\pi(f) = f(0)$.
- There is a natural translation action $\tau : G \to C_0(G/H)$ given by $\tau_g(k)(x + H) = k(x - g + H)$.
- There is a G-equivariant homomorphism $j : C_0(G/H) \to M(\mathrm{ind}_H^G(D, \alpha))$ determined by pointwise multiplication.

It is easy to check that these maps satisfy the following conditions:

i. For all $k \in C_0(G/H)$, $\pi(j(k)f) = k(H)\pi(f)$;
ii. for all $h \in H$, $\pi(\beta_h(f)) = \alpha_h(\pi(f))$;
iii. if $\pi(\beta_g(f)) = 0$ for all $g \in G$, then $f = 0$;
iv. and

$$\lim_{x+H\to\infty} \|f(x)\| = 0.$$

A characterization of the induced algebra

Theorem

Let H be a closed subgroup of G, let D be a C^*-algebra, and let $\alpha : H \to \operatorname{Aut} D$ be a strongly continuous action. Let E be a C^*-algebra, $\rho : E \to D$ a surjective homomorphism, $\gamma : G \to \operatorname{Aut}(E)$ a strongly continuous action and let $i : C_0(G/H) \to Z(M(E))$ be a G-equivariant homomorphism. Suppose that

- i *For all $k \in C_0(G/H)$ and $e \in E$, $\rho(i(k)e) = k(H)\rho(e)$;*
- ii *For all $h \in H$ and $e \in E$, $\rho(\gamma_h(e)) = \alpha_h(\rho(e))$;*
- iii *If $\rho(\gamma_g(e)) = 0$ for all $g \in G$, then $e = 0$;*
- iv *For all $e \in E$,*

$$\lim_{x+H\to\infty} \|\rho(\gamma_x(e))\| = 0.$$

Then there is a (unique) G-equivariant isomorphism $\Psi : E \to \operatorname{ind}_H^G(D, \alpha)$ such that: $\pi \circ \Psi = \rho$ and $j(k)\Psi(e) = \Psi(i(k)e)$ for all $k \in C_0(G/H)$ and $e \in E$.

Amenability

Theorem

Assume that (Σ, β) is a closed subgroupoid of an amenable groupoid (Γ, λ) such that $\Sigma^0 = \Gamma^0$. Then the map $j : C^*(\Sigma) \to M(C^*(\Gamma))$ defined for $a \in C_c(\Sigma)$ and $f \in C_c(\Gamma)$ via

$$(j(a)(f))(\gamma) = \int_\Sigma a(\eta) f(\eta^{-1}\gamma)\, d\,\beta^{r(\gamma)}(\eta)$$

is faithful.

Example

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- $C^*(\Gamma_{\mathcal{U}})$ is a continuous trace algebra with Prim $C^*(\Gamma_{\mathcal{U}}) = X$ and trivial Dixmier-Douady invariant $\delta(C^*(\Gamma_{\mathcal{U}})) = 0$.
- Let ϕ be the 1-cocycle defined by a Čech 1-cocycle λ_{ij}
- Let $\alpha : \mathbb{Z} \to \operatorname{Aut} C^*(\Gamma_{\mathcal{U}})$ be the action determined by ϕ.
- Then α can be shown to be locally unitary.

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- Let ϕ be the 1-cocycle defined by a Čech 1-cocycle λ_{ij}
- Let $\alpha : \mathbb{Z} \to \operatorname{Aut} C^*(\Gamma_{\mathcal{U}})$ be the action determined by ϕ.
- Then α can be shown to be locally unitary.
- Our theorem implies that $C^*(\Sigma_\phi) \cong \operatorname{ind}_{\mathbb{Z}}^{\mathbb{R}}(C^*(\Gamma_{\mathcal{U}}), \alpha)$.
- From work of Rosenberg and Raeburn it follows that $C^*(\Sigma_\phi)$ is a continuous trace algebra, Prim $C^*(\Sigma_\phi) \cong \mathbb{T} \times X$ and

$$\delta(C^*(\Sigma_\phi)) = z \times \eta(\alpha) = z \times [\lambda^\star] \in \check{H}^3(\mathbb{T} \times X, \mathbb{Z})$$

where z is the standard generator of $\check{H}^1(\mathbb{T}, \mathbb{Z})$,
$[\lambda^\star] \in \check{H}^2(X, \mathbb{Z}) = [\lambda] \in \check{H}^1(X, \mathcal{T})$, and $\eta(\alpha)$ is the Phillips-Raeburn obstruction.