Hausdorff Dimension of Kuperberg Minimal Sets

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Introduction

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- This flow preserves a nontrivial minimal set with a fractal structure.
- We use tools from conformal iterated function systems and thermodynamic formalism to calculate the Hausdorff dimension of this minimal set.

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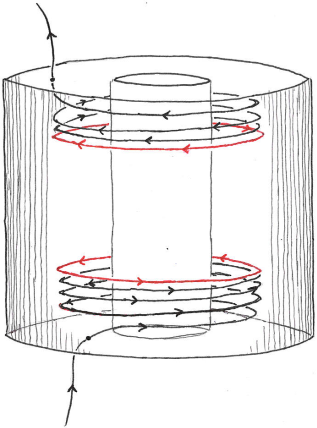
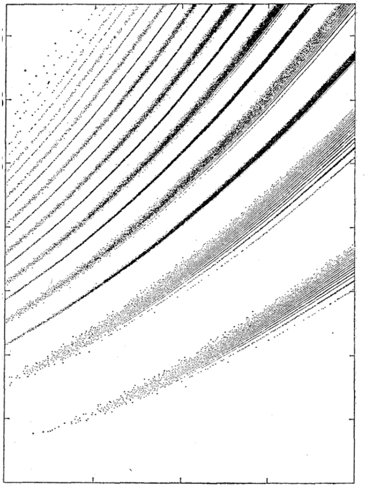
- Kuperberg 1994: Constructed a C^∞ counterexample.
  - Modified Wilson's plug by *self-insertion*

Wilson's minimal set



Two periodic orbits

Kuperberg's minimal set



Cross-section of Kuperberg minimal set

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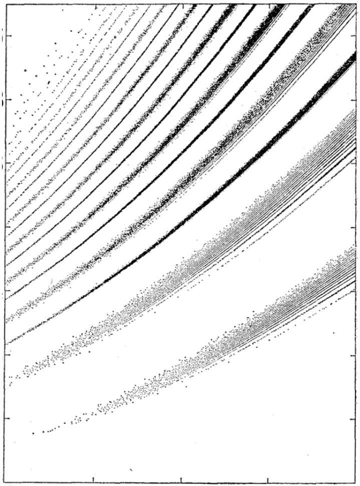

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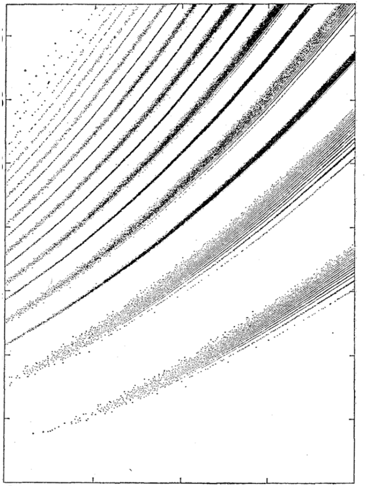

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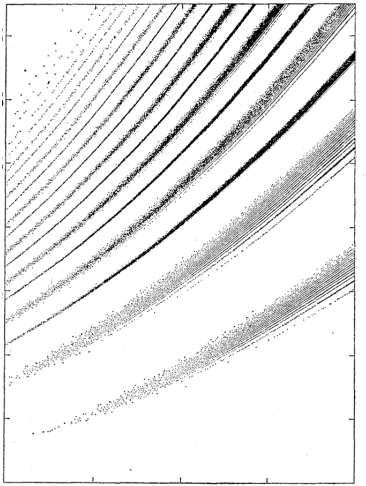

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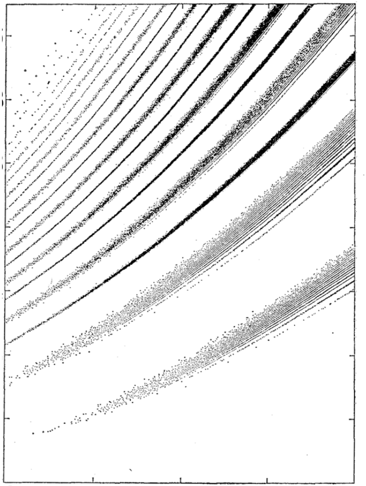

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- This IFS is defined in terms of a *pseudogroup* of first-return maps to a section of the flow.

Iterated Function Systems

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- Necessary to assume that ϕ_i are conformal (CIFS).

Topological Pressure of a CIFS

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- Mauldin and Urbański (1996) extended this formalism to countable alphabets I.

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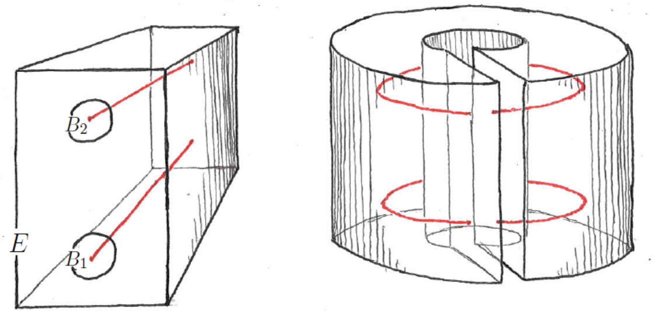

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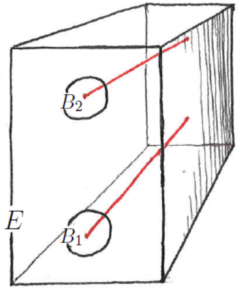

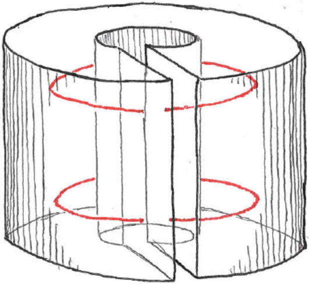

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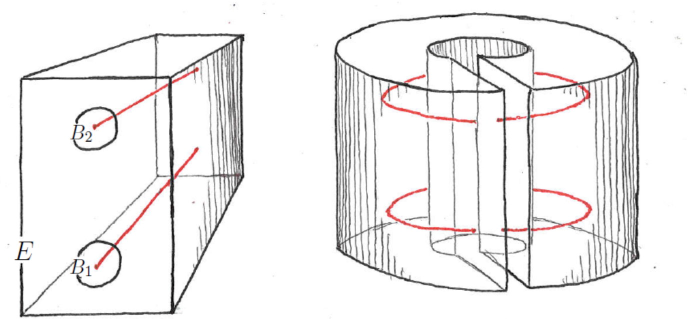

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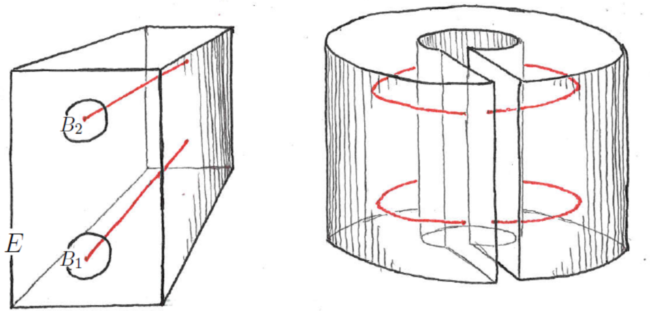

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- g decays to zero inside each B_i, at $r = 2$.

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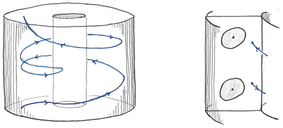

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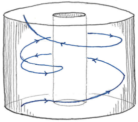

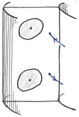

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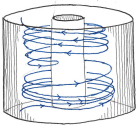

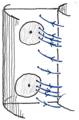

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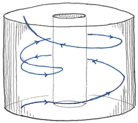

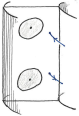

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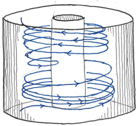

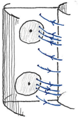

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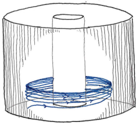

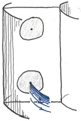

The Wilson pseudogroup

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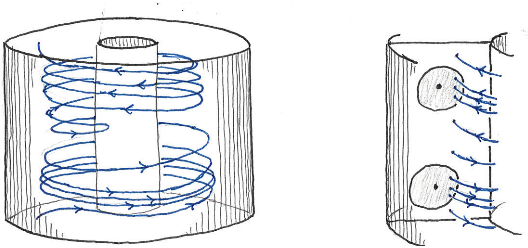

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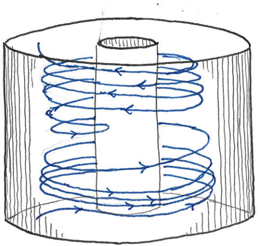

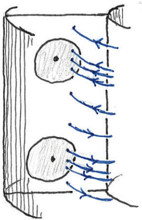

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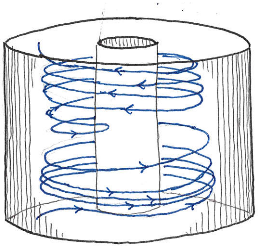

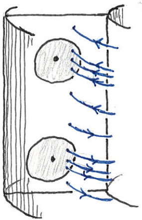

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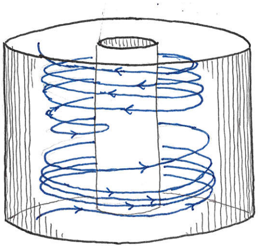

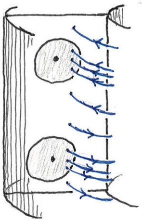

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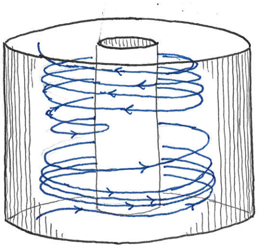

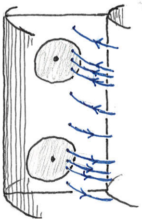

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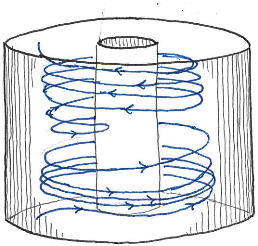
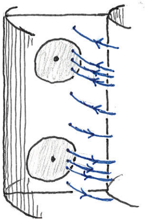
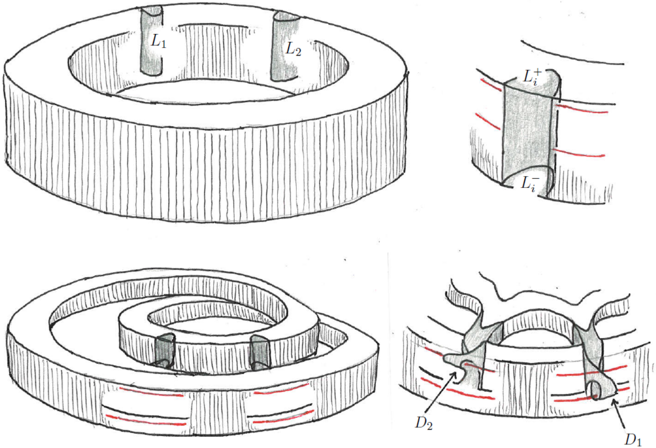




- E is not a global section for ϕ_t.
- Φ is not defined on all of E.
- Φ is not continuous everywhere it is defined.
- Φ^n is not defined for every n, even when Φ is defined.
- Φ generates a *pseudogroup* which reflects dynamics of ϕ_t.

Kuperberg's plug

To construct a plug with no periodic orbits, Kuperberg inserted the Wilson plug into itself. The resulting plug K inherits a vector field $\mathcal{K}$ with flow ψ_t.



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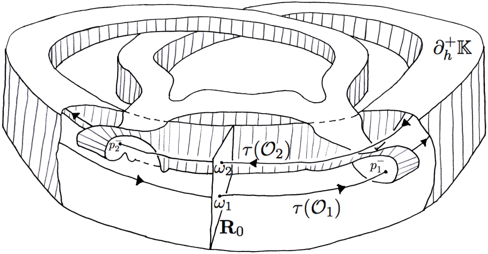

- As with Wilson pseudogroup Φ, we obtain the Kuperberg pseudogroup generated by $\Psi : \mathbf{R}_0 \to \mathbf{R}_0$.

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Kuperberg (1994):

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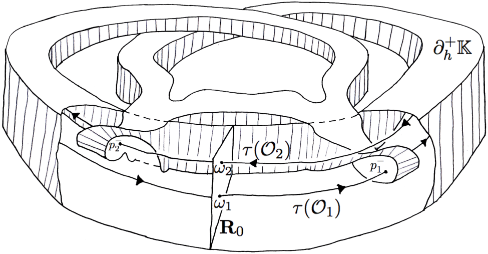

- As with Wilson pseudogroup Φ, we obtain the Kuperberg pseudogroup generated by $\Psi : \mathbf{R}_0 \to \mathbf{R}_0$.
- Ψ is generated by three maps: the Wilson return map Φ, as well as the insertion maps σ_1 and σ_2.

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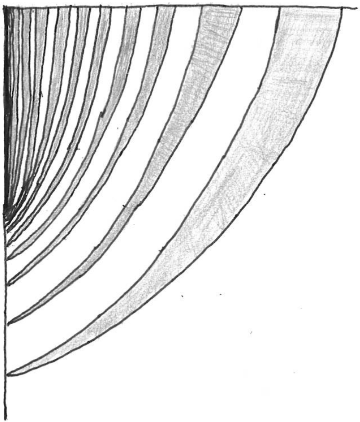
- The second characterization allows us to stratify the minimal set into *propellers* corresponding to each *level* of insertion.

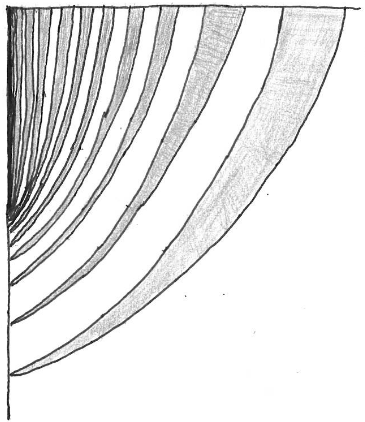
$$\mathcal{M} = \bigcup_{n=1}^{\infty} \mathcal{M}_n$$

Level-one propeller P_1



Cross-section of P_1

Level-one propeller P_1



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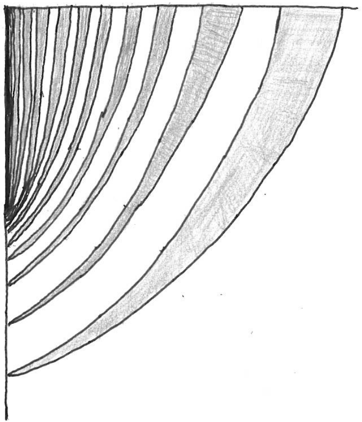

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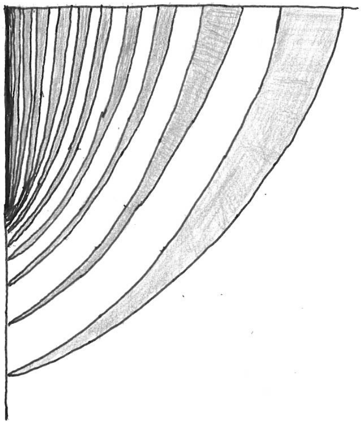

Cross-section of P_1

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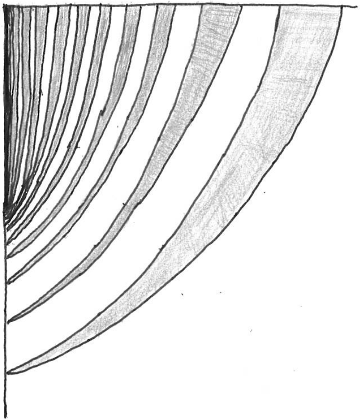
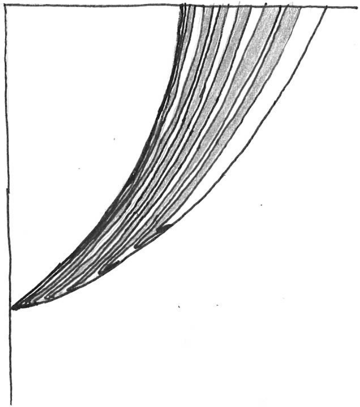


Cross-section of P_1

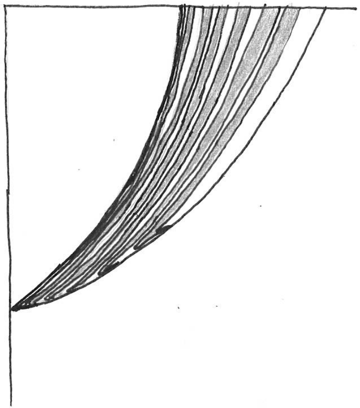
- Curves are images under powers of Φ, the generator of the Wilson pseudogroup
- Propeller P_1 bounds a closed region A_1.
- The pseudogroup Ψ contracts P_1 in the radial direction.
- Infinite returns of the propeller implies symbolic dynamics on an infinite alphabet.

Level-two propeller P_2



Cross-section of P_2

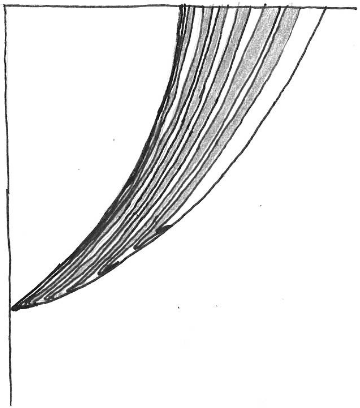
Level-two propeller P_2



Cross-section of P_2

- Curves are images under one insertion σ of powers of Φ.

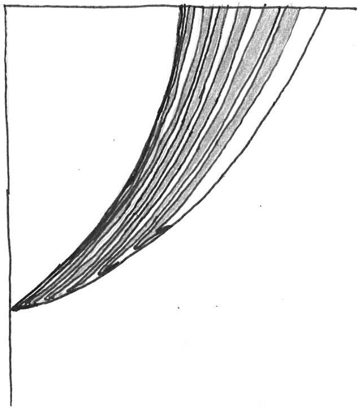
Level-two propeller P_2



Cross-section of P_2

- Curves are images under one insertion σ of powers of Φ.
- Propeller P_2 bounds a family of closed regions $A_{2,i}$.

Level-two propeller P_2



Cross-section of P_2

- Curves are images under one insertion σ of powers of Φ.
- Propeller P_2 bounds a family of closed regions $A_{2,i}$.
- Nesting property: $A_{2,i} \subset A_i$

Dimension estimates on $\mathcal{M}$

Theorem (I.)

Let $C \subset [0,1]$ be the transverse Cantor set of $\mathcal{M}$.

- There exists a CIFS on $[0,1]$ with limit set C.
- $s = \dim_H(C)$ is the unique root of a dynamically defined pressure function.
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Corollary: $2.5877 \leq \dim_H(\mathcal{M}) \leq \ 2.8643$.

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