Hilbert C^*-Modules over Groupoid Dynamical Systems and Square-Integrable Representations

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Summary of Talk

We first review the definitions and properties of Banach bundles and C^*-bundles.

We then introduce the concept of a bundle of Hilbert C^*-modules over a C^*-bundle.

After developing enough machinery concerning such bundles of Hilbert C^*-modules, we describe how to use them to construct a theory of square-integrable representations of groupoid dynamical systems.

Definition (Banach Bundles [Fell & Doran, 1988])

Let X be a locally compact Hausdorff space. A Banach bundle over X is a pair $(\mathscr{B}, p)$ with the following properties:

- $\mathscr{B}$ is a topological space, and p is a continuous open surjection from $\mathscr{B}$ onto X.
- For each $x \in X$, the pre-image $p^{\leftarrow}[\{x\}]$ of x via p, which we denote by $\mathscr{B}_x$ and call the fiber of $\mathscr{B}$ over x, has the structure of a Banach space.
- The map $\begin{Bmatrix} \mathscr{B} & \to & \mathbb{R} \\ b & \mapsto & \|b\|_{p(b)} \end{Bmatrix}$ is upper semicontinuous.
- The map $\begin{Bmatrix} \mathscr{B}^{(2)} & \to & \mathscr{B} \\ (b_1, b_2) & \mapsto & b_1 + b_2 \end{Bmatrix}$ is continuous, where
$$\mathscr{B}^{(2)} := \{(b_1, b_2) \in \mathscr{B} \times \mathscr{B} \mid p(b_1) = p(b_2)\}.$$
- For each $\lambda \in \mathbb{C}$, the map $\begin{Bmatrix} \mathscr{B} & \to & \mathscr{B} \\ b & \mapsto & \lambda \cdot b \end{Bmatrix}$ is continuous.
- If $(b_i)_{i \in I}$ is a net in $\mathscr{B}$ such that $\|b_i\|_{p(b_i)} \to 0$ and $p(b_i) \to x$ for some $x \in X$, then $b_i \to 0_x$, where 0_x denotes the zero element of the Banach space $\mathscr{B}_x$.

C^*- Bundles

Definition (C^*-Bundles [Williams, 2007])

A C^*-bundle over a locally compact Hausdorff space X is just a Banach bundle $(\mathscr{A}, p)$ over X with the following additional properties:

- For each $x \in X$, the fiber $\mathscr{A}_x$ is a C^*-algebra.
- The map $\left\{ \begin{matrix} \mathscr{A} & \to & \mathscr{A} \\ a & \mapsto & a^* \end{matrix} \right\}$ is continuous.
- The map $\left\{ \begin{matrix} \mathscr{A}^{(2)} & \to & \mathscr{A} \\ (a_1, a_2) & \mapsto & a_1 a_2 \end{matrix} \right\}$ is continuous.

Given a C^*-bundle $(\mathscr{A}, p)$ over a locally compact Hausdorff space X, the set $\Gamma_0(\mathscr{A}, p)$ of continuous sections of $(\mathscr{A}, p)$ that vanish at infinity is a C^*-algebra.

In fact, $\Gamma_0(\mathscr{A}, p)$ is a $C_0(X)$-algebra, i.e., there is a non-degenerate $*$-homomorphism from $C_0(X)$ to the center of the multiplier algebra of $\Gamma_0(\mathscr{A}, p)$.

Given a locally compact Hausdorff space X, it is a deep result that any $C_0(X)$-algebra is $*$-isomorphic to $\Gamma_0(\mathscr{A}, p)$ for some C^*-bundle $(\mathscr{A}, p)$ over X.

Bundles of Hilbert C^*-Modules

From now on, $(\mathscr{A}, p)$ denotes a C^*-bundle over a locally compact Hausdorff space X.

Definition (Bundles of Hilbert C^*-Modules (H., 2017))

An $(\mathscr{A}, p)$-bundle of Hilbert C^*-modules over X is defined as a Banach bundle $(\mathscr{E}, q)$ over X with the following additional properties:

- For each $x \in X$, the fiber $\mathscr{E}_x$ is a Hilbert $\mathscr{A}_x$-module.
- The map $\left\{ \begin{matrix} \mathscr{E} \;{}_q\!\times_p \mathscr{A} & \to & \mathscr{E} \\ (\zeta, a) & \mapsto & \zeta \bullet a \end{matrix} \right\}$ is continuous, where
$$\mathscr{E} \;{}_q\!\times_p \mathscr{A} := \{(\zeta, a) \in \mathscr{E} \times \mathscr{A} \mid q(\zeta) = p(a)\}.$$
- The map $\left\{ \begin{matrix} \mathscr{E}^{(2)} & \to & \mathscr{A} \\ (\zeta, \eta) & \mapsto & \langle \zeta, \eta \rangle_{\mathscr{E}_{p(\zeta)}} \end{matrix} \right\}$ is continuous.

Given an $(\mathscr{A}, p)$-bundle $(\mathscr{E}, q)$ of Hilbert C^*-modules over X, it can be shown that the set $\Gamma_0(\mathscr{E}, q)$ of continuous sections of $(\mathscr{E}, q)$ that vanish at infinity is actually a Hilbert $\Gamma_0(\mathscr{A}, p)$-module.

Question: Is the converse true?

- Let $\mathcal{E}$ be a Hilbert $\Gamma_0(\mathscr{A}, p)$-module.
- For each $x \in X$, let I_x denote the set of sections in $\Gamma_0(\mathscr{A}, p)$ that vanish at x, i.e., for any $\sigma \in I_x$, we have $\sigma(x) = 0_x$.
- Then I_x is an ideal of $\Gamma_0(\mathscr{A}, p)$, and $\mathcal{E}_x := \left\{ \zeta \in \mathcal{E} \;\middle|\; \langle \zeta, \zeta \rangle_{\mathcal{E}} \in I_x \right\}$ is a closed linear subspace of $\mathcal{E}$ that is easily shown to be a Hilbert I_x-module.
- Hence, the quotient space $\mathcal{E}/\mathcal{E}_x$ is a Hilbert $\mathscr{A}_x$-module (as $\Gamma_0(\mathscr{A}, p)/I_x \cong \mathscr{A}_x$).
- Form the disjoint union $\mathscr{E} := \displaystyle\bigsqcup_{x \in X} \mathcal{E}/\mathcal{E}_x$, and define $q : \mathscr{E} \to X$ by $q(\zeta + \mathcal{E}_x) := x$ for all $\zeta \in \mathcal{E}$ and $x \in X$.
- Then $(\mathscr{E}, q)$ is a bundle of Hilbert C^*-modules over X, but with no topology yet.
- Now, $\Delta := \left\{ (\zeta + \mathcal{E}_x)_{x \in X} \;\middle|\; \zeta \in \mathcal{E} \right\}$ is a collection of sections of $(\mathscr{E}, q)$ such that:
  - The map $\begin{Bmatrix} X & \to & \mathbb{R}_{\geq 0} \\ x & \mapsto & \|\sigma(x)\|_{\mathscr{E}_x} \end{Bmatrix}$ is upper semicontinuous for each $\sigma \in \Delta$.
  - For each $x \in X$, $\{\sigma(x) \mid \sigma \in \Delta\}$ is a dense subset (in fact, all) of $\mathscr{E}_x$.
- By a theorem of Fell, one can define a unique topology on $\mathscr{E}$ so that each section in Δ is continuous. Furthermore, under this topology, $\Delta = \Gamma_0(\mathscr{E}, q)$, which can be proven using a partition-of-unity argument found in [Williams, 2007].

Finally, one verifies that $(\mathscr{E}, q)$ is an $(\mathscr{A}, p)$-bundle of Hilbert C^*-modules over X.

Proposition (H., 2017)

$\mathcal{E}$ and $\Gamma_0(\mathcal{E}, q)$ are unitarily isomorphic as Hilbert $\Gamma_0(\mathscr{A}, p)$-modules.

Proof.

By a result of E. Lance, it suffices to show that there exists an isometric and surjective $\Gamma_0(\mathscr{A}, p)$-linear map from $\mathcal{E}$ to $\Gamma_0(\mathscr{E}, q)$.

Define a $\Gamma_0(\mathscr{A}, p)$-linear map $S : \mathcal{E} \to \Gamma_0(\mathscr{E}, q)$ by $S(\zeta) := (\zeta + \mathcal{E}_x)_{x \in X}$. Then

$$\begin{aligned}
\left\| (\zeta + \mathcal{E}_x)_{x \in X} \right\|_{\Gamma_0(\mathscr{E}, q)} &= \sup_{x \in X} \| \zeta + \mathcal{E}_x \|_{\mathscr{E}_x} \\
&= \sup_{x \in X} \left\| \langle \zeta, \zeta \rangle_{\mathcal{E}}(x) \right\|_{\mathscr{A}_x}^{1/2} \\
&= \left\| \langle \zeta, \zeta \rangle_{\mathcal{E}} \right\|_{\Gamma_0(\mathscr{A}, p)}^{1/2} \\
&= \| \zeta \|_{\mathcal{E}}. \qquad \text{(Therefore, } S \text{ is isometric.)}
\end{aligned}$$

The surjectivity of S follows from the earlier comment that $\Delta = \Gamma_0(\mathscr{E}, q)$. □

An Equivalence of Categories

The class of $(\mathscr{A}, p)$-bundles of Hilbert C^*-modules over X forms a category.

Given two objects $(\mathscr{E}, q)$ and $(\mathscr{F}, r)$ of this class, a morphism from $(\mathscr{E}, q)$ to $(\mathscr{F}, r)$ is a (continuous) bundle map $\mathscr{T} : (\mathscr{E}, q) \to (\mathscr{F}, r)$ that satisfies the following conditions:

- $\mathscr{T}_x := \mathscr{T}|_{\mathscr{E}_x} : \mathscr{E}_x \to \mathscr{F}_x$ is an adjointable operator between Hilbert $\mathscr{A}_x$-modules.
- $\sup\limits_{x \in X} \|\mathscr{T}_x\|_{\mathscr{E}_x \to \mathscr{F}_x} < \infty$.

If $\mathcal{E}$ and $\mathcal{F}$ are Hilbert $\Gamma_0(\mathscr{A}, p)$-modules, and if $T : \mathcal{E} \to \mathcal{F}$ is an adjointable operator, then it is not difficult to show that the $\Gamma_0(\mathscr{A}, p)$-linearity of T give us a morphism $\mathscr{T}$ between the $(\mathscr{A}, p)$-bundles of Hilbert C^*-modules corresponding to $\mathcal{E}$ and $\mathcal{F}$.

This observation leads us the following categorical result.

Proposition (H., 2017)

There is a natural isomorphism $\mathfrak{F}$ from the category of Hilbert $\Gamma_0(\mathscr{A}, p)$-modules (with morphisms given by the usual adjointable operators) to the category of $(\mathscr{A}, p)$-bundles of Hilbert C^*-modules over X (with morphisms as described above).

Definition (Groupoids)

A groupoid is a small category $\mathcal{G}$ in which every morphism is an isomorphism. The set of objects of $\mathcal{G}$ is called the unit space of $\mathcal{G}$ and is denoted by $\mathcal{G}^{(0)}$. One also has maps $s, t : \mathcal{G} \to \mathcal{G}^{(0)}$ that take each morphism to its source and target respectively.

Definition (Groupoid Dynamical Systems)

A groupoid dynamical system is a quadruple $(\mathcal{G}, \alpha, \mathscr{A}, p)$ with the following properties:

- $\mathcal{G}$ is a locally compact Hausdorff groupoid (this means that $\mathcal{G}^{(0)}$ is Hausdorff, the groupoid operations (including s and t) are continuous, and s and t are open).
- $(\mathscr{A}, p)$ is a C^*-bundle over $\mathcal{G}^{(0)}$.
- α is a $\mathcal{G}$-indexed family of C^*-algebraic isomorphisms governed by the following:
  - $\alpha_\gamma : \mathscr{A}_{s(\gamma)} \to \mathscr{A}_{t(\gamma)}$ for each $\gamma \in \mathcal{G}$.
  - $\alpha_{\gamma_1} \circ \alpha_{\gamma_2} = \alpha_{\gamma_1 \gamma_2}$ for each $(\gamma_1, \gamma_2) \in \mathcal{G}^{(2)} := \mathcal{G} \ {}_s\!\times_t \mathcal{G}$.
  - The map $\left\{ \begin{matrix} \mathcal{G} \ {}_s\!\times_p \mathscr{A} & \to & \mathscr{A} \\ (\gamma, a) & \mapsto & \alpha_\gamma(a) \end{matrix} \right\}$ is continuous.

From now on, $(\mathcal{G}, \alpha, \mathscr{A}, p)$ denotes a groupoid dynamical system.

Definition (Hilbert C^*-Modules over a Groupoid Dynamical System (H., 2017))

A Hilbert $(\mathcal{G}, \alpha, \mathscr{A}, p)$-module is a pair $(\mathcal{E}, \Phi)$ with the following properties:

- $\mathcal{E}$ is a Hilbert $\Gamma_0(\mathscr{A}, p)$-module (where we let $(\mathscr{E}, q) := \mathfrak{F}(\mathcal{E})$).
- Φ is a groupoid homomorphism from $\mathcal{G}$ to
$$\mathrm{Iso}(\mathscr{E}, q) := \left\{ (t(\gamma), T, s(\gamma)) \;\middle|\; \gamma \in \mathcal{G} \text{ and } T \in \mathrm{Isom}\left(\mathscr{E}_{s(\gamma)}, \mathscr{E}_{t(\gamma)}\right) \right\}$$
such that for each $\gamma \in \mathcal{G}$, there is a linear isometry $\Phi_\gamma \in \mathrm{Isom}\left(\mathscr{E}_{s(\gamma)}, \mathscr{E}_{t(\gamma)}\right)$ giving
$$\Phi(\gamma) = (t(\gamma), \Phi_\gamma, s(\gamma)).$$
- The map $\left\{ \begin{matrix} \mathcal{G} \;{}_s{\times}_q\; \mathscr{E} & \to & \mathscr{E} \\ (\gamma, \zeta) & \mapsto & \Phi_\gamma(\zeta) \end{matrix} \right\}$ is continuous.
- $\Phi(x) = (x, \mathrm{Id}_{\mathscr{E}_x}, x)$ for each $x \in \mathcal{G}^{(0)}$.
- $\Phi_\gamma\left(\zeta \bullet_{s(\gamma)} a\right) = \Phi_\gamma(\zeta) \bullet_{t(\gamma)} \alpha_\gamma(a)$ for each $\gamma \in \mathcal{G}$, $\zeta \in \mathscr{E}_{s(\gamma)}$ and $a \in \mathscr{A}_{s(\gamma)}$.
- $\langle \Phi_\gamma(\zeta), \Phi_\gamma(\eta) \rangle_{\mathscr{E}_{t(\gamma)}} = \alpha_\gamma\left(\langle \zeta, \eta \rangle_{\mathscr{E}_{s(\gamma)}} \right)$ for each $\gamma \in \mathcal{G}$ and $\zeta, \eta \in \mathscr{E}_{s(\gamma)}$.

The class of Hilbert $(\mathcal{G}, \alpha, \mathscr{A}, p)$-modules forms a category.

Given two objects $(\mathcal{E}, \Phi)$ and $(\mathcal{F}, \Psi)$ of this class, a morphism from $(\mathcal{E}, \Phi)$ to $(\mathcal{F}, \Psi)$ is an adjointable operator $T : \mathcal{E} \to \mathcal{F}$ that satisfies the following equivariance property: If $\mathscr{T} := \mathfrak{F}(T)$, then $\mathscr{T}_{t(\gamma)} \circ \Phi_\gamma = \Psi_\gamma \circ \mathscr{T}_{s(\gamma)}$ for each $\gamma \in \mathcal{G}$.

The equivariance property means that if $(\mathscr{E}, q) := \mathfrak{F}(\mathcal{E})$ and $(\mathscr{F}, r) := \mathfrak{F}(\mathcal{F})$, then the following diagram commutes:

$$\begin{array}{ccc}
\mathscr{E}_{s(\gamma)} & \xrightarrow{\mathscr{T}_{s(\gamma)}} & \mathscr{F}_{s(\gamma)} \\
\Phi_\gamma \big\downarrow & & \big\downarrow \Psi_\gamma \\
\mathscr{E}_{t(\gamma)} & \xrightarrow[\mathscr{T}_{t(\gamma)}]{} & \mathscr{F}_{t(\gamma)}
\end{array}$$

Examples of Hilbert $(\mathcal{G},\alpha,\mathscr{A},p)$-Modules

Example (An elementary example)

A simple non-trivial example of a Hilbert $(\mathcal{G},\alpha,\mathscr{A},p)$-module is $(\Gamma_0(\mathscr{A},p),\alpha)$, where we make use of the isomorphism $\Gamma_0(\mathscr{A},p)/I_x \cong \mathscr{A}_x$ for all $x \in \mathcal{G}^{(0)}$.

We now suppose that the groupoid $\mathcal{G}$ has a left Haar system $(\lambda^x)_{x\in\mathcal{G}^{(0)}}$ of measures.

Example (The standard Hilbert $(\mathcal{G},\alpha,\mathscr{A},p)$-module (H., 2017))

Let $(t^*\mathscr{A},t^*p)$ denote the C^*-bundle over $\mathcal{G}$ formed by pulling back the bundle $(\mathscr{A},p)$ over $\mathcal{G}^{(0)}$ under t. Complete $\Gamma_c(t^*\mathscr{A},t^*p)$ to a Hilbert $\Gamma_0(\mathscr{A},p)$-module $\Gamma^2(t^*\mathscr{A},t^*p)$. The corresponding $(\mathscr{A},p)$-bundle of Hilbert C^*-modules has the property that its fiber over each $x \in \mathcal{G}^{(0)}$ is unitarily isomorphic to the Hilbert $\mathscr{A}_x$-module $L^2(\mathcal{G}^x,\mathscr{A}_x)$.

There is a groupoid homomorphism $\Theta : \mathcal{G} \to \operatorname{Iso}\left(\mathfrak{F}\left(\Gamma^2(t^*\mathscr{A},t^*p)\right)\right)$ such that

$$\forall \phi \in C_c\left(\mathcal{G}^{s(\gamma)},\mathscr{A}_{s(\gamma)}\right): \quad \Theta_\gamma\left(\iota_{s(\gamma)}(\phi)\right) = \iota_{t(\gamma)}\left(\left\{\begin{matrix} \mathcal{G}^{t(\gamma)} & \to & \mathscr{A}_{t(\gamma)} \\ \gamma' & \mapsto & \alpha_\gamma\left(\phi\left(\gamma^{-1}\gamma'\right)\right) \end{matrix}\right\}\right),$$

where ι_x denotes the canonical dense linear embedding of $C_c(\mathcal{G}^x,\mathscr{A}_x)$ into $L^2(\mathcal{G}^x,\mathscr{A}_x)$. The pair $\left(\Gamma^2(t^*\mathscr{A},t^*p),\Theta\right)$ is called the standard $(\mathcal{G},\alpha,\mathscr{A},p)$-module.

Definition (Bra-Ket Operators for Hilbert $(\mathcal{G}, \alpha, \mathscr{A}, p)$-Modules (H., 2017))

Let $(\mathcal{E}, \Phi)$ be a Hilbert $(\mathcal{G}, \alpha, \mathscr{A}, p)$-module with $(\mathscr{E}, q) := \mathfrak{F}(\mathcal{E})$. For any $\zeta \in \mathcal{E}$, define two linear operators

$$\langle\!\langle \zeta | : \Gamma_0(\mathscr{E}, q) \to \Gamma_b(t^* \mathscr{A}, t^* p) \qquad \text{and} \qquad | \zeta \rangle\!\rangle : \Gamma_c(t^* \mathscr{A}, t^* p) \to \Gamma_0(\mathscr{E}, q),$$

called the bra and ket of ζ respectively, by

$$\forall \sigma \in \Gamma_0(\mathscr{E}, q) : \qquad \langle\!\langle \zeta | (\sigma) := \left\{ \begin{matrix} \mathcal{G} & \to & t^* \mathscr{A} \\ \gamma & \mapsto & \left\langle \Phi_\gamma \left(\zeta + \mathcal{E}_{s(\zeta)} \right), \sigma(t(\gamma)) \right\rangle_{\mathscr{E}_{t(\gamma)}} \end{matrix} \right\};$$

$$\forall \phi \in \Gamma_c(t^* \mathscr{A}, t^* p) : \qquad | \zeta \rangle\!\rangle (\phi) := \left\{ \begin{matrix} X & \to & \mathscr{E} \\ x & \mapsto & \displaystyle\int_{\mathcal{G}} \Phi_\gamma(\zeta + \mathcal{E}_x) \bullet_x \phi(\gamma) \, \mathrm{d}\lambda^x(\gamma) \end{matrix} \right\}.$$

Given any $\zeta, \eta \in \mathcal{E}$, we have $\langle\!\langle \zeta | \eta \rangle\!\rangle [\Gamma_c(t^* \mathscr{A}, t^* p)] \subseteq \Gamma_b(t^* \mathscr{A}, t^* p)$.

Definition (Square-Integrable Vectors (H., 2017))

Let $(\mathcal{E}, \Phi)$ be a Hilbert $(\mathcal{G}, \alpha, \mathscr{A}, p)$-module with $(\mathscr{E}, q) := \mathfrak{F}(\mathcal{E})$. We then call $\zeta \in \mathcal{E}$ a square-integrable vector if and only if for any section $\sigma \in \Gamma_0(\mathscr{E}, q)$ and any net $(\varphi_i)_{i \in I}$ of functions in $C_c(\mathcal{G}, [0, 1])$ that converges uniformly to 1 on all compact subsets of $\mathcal{G}$, the corresponding net $(\iota(\varphi_i \cdot \langle\!\langle \zeta | (\sigma)))_{i \in I}$ in $\Gamma^2(t^* A, t^* p)$ is also convergent. The set of square-integrable vectors is denoted by $(\mathcal{E}, \Phi)_{\mathrm{si}}$.

We call $(\mathcal{E}, \Phi)$ a square-integrable representation of $(\mathcal{G}, \alpha, \mathscr{A}, p)$ if and only if $(\mathcal{E}, \Phi)_{\mathrm{si}}$ is dense in $\mathcal{E}$.

Using the concept of square-integrability, one can, in principle, achieve the following:

- A generalization of J. Brown's results on proper groupoid actions on C^*-algebras.
- The construction of generalized fixed-point algebras for proper groupoid actions on Hilbert C^*-modules.
- A classification of reduced groupoid crossed products.
- An equivariant version of Kasparov's Stabilization Theorem for groupoids.
- A definition of equivariant KK-theory for groupoid actions.

One only has to mimic the proofs given in [Meyer, 2001, Meyer, 2001] and [H, 2016].

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