Hilbert $C^*$-Modules over Groupoid Dynamical Systems and Square-Integrable Representations

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We first review the definitions and properties of Banach bundles and $C^*$-bundles.

We then introduce the concept of a bundle of Hilbert $C^*$-modules over a $C^*$-bundle.

After developing enough machinery concerning such bundles of Hilbert $C^*$-modules, we describe how to use them to construct a theory of square-integrable representations of groupoid dynamical systems.
Banach Bundles

Definition (Banach Bundles [Fell & Doran, 1988])

Let $X$ be a locally compact Hausdorff space. A Banach bundle over $X$ is a pair $(\mathcal{B}, p)$ with the following properties:

- $\mathcal{B}$ is a topological space, and $p$ is a continuous open surjection from $\mathcal{B}$ onto $X$.
- For each $x \in X$, the pre-image $p^{-1}([x])$ of $x$ via $p$, which we denote by $\mathcal{B}_x$ and call the fiber of $\mathcal{B}$ over $x$, has the structure of a Banach space.
- The map $\left\{ \begin{array}{c} \mathcal{B} \\ b \end{array} \rightarrow \mathbb{R} \right\} \rightarrow \| b \|_{p(b)}$ is upper semicontinuous.
- The map $\left\{ \begin{array}{c} \mathcal{B}^{(2)} \\ (b_1, b_2) \end{array} \rightarrow \mathcal{B} \right\} \rightarrow b_1 + b_2$ is continuous, where

  $\mathcal{B}^{(2)} := \{(b_1, b_2) \in \mathcal{B} \times \mathcal{B} \mid p(b_1) = p(b_2)\}$.

- For each $\lambda \in \mathbb{C}$, the map $\left\{ \begin{array}{c} \mathcal{B} \\ b \end{array} \rightarrow \mathcal{B} \right\} \rightarrow \lambda \cdot b$ is continuous.
- If $(b_i)_{i \in I}$ is a net in $\mathcal{B}$ such that $\| b_i \|_{p(b_i)} \rightarrow 0$ and $p(b_i) \rightarrow x$ for some $x \in X$, then $b_i \rightarrow 0_x$, where $0_x$ denotes the zero element of the Banach space $\mathcal{B}_x$. 
A $C^*$-bundle over a locally compact Hausdorff space $X$ is just a Banach bundle $(\mathcal{A}, p)$ over $X$ with the following additional properties:

- For each $x \in X$, the fiber $\mathcal{A}_x$ is a $C^*$-algebra.
- The map $\left\{ \mathcal{A} \to \mathcal{A}^* \right\}$ is continuous.
- The map $\left\{ \mathcal{A}^{(2)} \to \mathcal{A} \; (a_1, a_2) \mapsto a_1a_2 \right\}$ is continuous.

Given a $C^*$-bundle $(\mathcal{A}, p)$ over a locally compact Hausdorff space $X$, the set $\Gamma_0(\mathcal{A}, p)$ of continuous sections of $(\mathcal{A}, p)$ that vanish at infinity is a $C^*$-algebra.

In fact, $\Gamma_0(\mathcal{A}, p)$ is a $C_0(X)$-algebra, i.e., there is a non-degenerate $*$-homomorphism from $C_0(X)$ to the center of the multiplier algebra of $\Gamma_0(\mathcal{A}, p)$.

Given a locally compact Hausdorff space $X$, it is a deep result that any $C_0(X)$-algebra is $*$-isomorphic to $\Gamma_0(\mathcal{A}, p)$ for some $C^*$-bundle $(\mathcal{A}, p)$ over $X$. 
From now on, \((\mathcal{A}, p)\) denotes a \(C^*\)-bundle over a locally compact Hausdorff space \(X\).

**Definition (Bundles of Hilbert \(C^*\)-Modules (H., 2017))**

An \((\mathcal{A}, p)\)-bundle of Hilbert \(C^*\)-modules over \(X\) is defined as a Banach bundle \((\mathcal{E}, q)\) over \(X\) with the following additional properties:

- For each \(x \in X\), the fiber \(\mathcal{E}_x\) is a Hilbert \(\mathcal{A}_x\)-module.
- The map \(\mathcal{E}_{q \times_p \mathcal{A}} \to \mathcal{E} \quad (\zeta, a) \mapsto \zeta \cdot a\) is continuous, where \(\mathcal{E}_{q \times_p \mathcal{A}} := \{(\zeta, a) \in \mathcal{E} \times \mathcal{A} \mid q(\zeta) = p(a)\}\).
- The map \(\mathcal{E}^{(2)} \to \mathcal{A} \quad (\zeta, \eta) \mapsto \langle \zeta, \eta \rangle_{\mathcal{E}_{q(\zeta)}}\) is continuous.

Given an \((\mathcal{A}, p)\)-bundle \((\mathcal{E}, q)\) of Hilbert \(C^*\)-modules over \(X\), it can be shown that the set \(\Gamma_0(\mathcal{E}, q)\) of continuous sections of \((\mathcal{E}, q)\) that vanish at infinity is actually a Hilbert \(\Gamma_0(\mathcal{A}, p)\)-module.

**Question:** Is the converse true?
Let $\mathcal{E}$ be a Hilbert $\Gamma_0(\mathcal{A}, p)$-module.

For each $x \in X$, let $I_x$ denote the set of sections in $\Gamma_0(\mathcal{A}, p)$ that vanish at $x$, i.e., for any $\sigma \in I_x$, we have $\sigma(x) = 0$.

Then $I_x$ is an ideal of $\Gamma_0(\mathcal{A}, p)$, and $\mathcal{E}_x := \{ \zeta \in \mathcal{E} \mid \langle \zeta, \zeta \rangle_{\mathcal{E}} \in I_x \}$ is a closed linear subspace of $\mathcal{E}$ that is easily shown to be a Hilbert $I_x$-module.

Hence, the quotient space $\mathcal{E}/\mathcal{E}_x$ is a Hilbert $\mathcal{A}_x$-module (as $\Gamma_0(\mathcal{A}, p)/I_x \cong \mathcal{A}_x$).

Form the disjoint union $\mathcal{E} := \bigsqcup_{x \in X} \mathcal{E}/\mathcal{E}_x$, and define $q : \mathcal{E} \to X$ by $q(\zeta + \mathcal{E}_x) := x$ for all $\zeta \in \mathcal{E}$ and $x \in X$.

Then $(\mathcal{E}, q)$ is a bundle of Hilbert $C^*$-modules over $X$, but with no topology yet.

Now, $\Delta := \{ (\zeta + \mathcal{E}_x)_{x \in X} \mid \zeta \in \mathcal{E} \}$ is a collection of sections of $(\mathcal{E}, q)$ such that:

- The map $\left\{ \begin{array}{ccc} X & \to & \mathbb{R}^+ \\ x & \mapsto & \| \sigma(x) \|_{\mathcal{E}_x} \end{array} \right\}$ is upper semicontinuous for each $\sigma \in \Delta$.

- For each $x \in X$, $\{ \sigma(x) \mid \sigma \in \Delta \}$ is a dense subset (in fact, all) of $\mathcal{E}_x$.

By a theorem of Fell, one can define a unique topology on $\mathcal{E}$ so that each section in $\Delta$ is continuous. Furthermore, under this topology, $\Delta = \Gamma_0(\mathcal{E}, q)$, which can be proven using a partition-of-unity argument found in [Williams, 2007].
Finally, one verifies that \((\mathcal{E}, q)\) is an \((\mathcal{A}, p)\)-bundle of Hilbert \(C^*\)-modules over \(X\).

**Proposition (H., 2017)**

\(\mathcal{E}\) and \(\Gamma_0(\mathcal{E}, q)\) are unitarily isomorphic as Hilbert \(\Gamma_0(\mathcal{A}, p)\)-modules.

**Proof.**

By a result of E. Lance, it suffices to show that there exists an isometric and surjective \(\Gamma_0(\mathcal{A}, p)\)-linear map from \(\mathcal{E}\) to \(\Gamma_0(\mathcal{E}, q)\).

Define a \(\Gamma_0(\mathcal{A}, p)\)-linear map \(S : \mathcal{E} \to \Gamma_0(\mathcal{E}, q)\) by \(S(\zeta) := (\zeta + \mathcal{E}_x)_{x \in X}\). Then

\[
\|(\zeta + \mathcal{E}_x)_{x \in X}\|_{\Gamma_0(\mathcal{E}, q)} = \sup_{x \in X} \|\zeta + \mathcal{E}_x\|_{\mathcal{E}_x}
\]

\[
= \sup_{x \in X} \|\langle \zeta, \zeta \rangle_{\mathcal{E}(x)}\|_{\mathcal{A}_x}^{1/2}
\]

\[
= \|\langle \zeta, \zeta \rangle_{\mathcal{E}}\|_{\Gamma_0(\mathcal{A}, p)}^{1/2}
\]

\[
= \|\zeta\|_{\mathcal{E}}. \quad \text{(Therefore, \(S\) is isometric.)}
\]

The surjectivity of \(S\) follows from the earlier comment that \(\Delta = \Gamma_0(\mathcal{E}, q)\).
An Equivalence of Categories

The class of \((\mathcal{A}, p)\)-bundles of Hilbert \(C^*\)-modules over \(X\) forms a category.

Given two objects \((\mathcal{E}, q)\) and \((\mathcal{F}, r)\) of this class, a morphism from \((\mathcal{E}, q)\) to \((\mathcal{F}, r)\) is a (continuous) bundle map \(T : (\mathcal{E}, q) \rightarrow (\mathcal{F}, r)\) that satisfies the following conditions:

- \(T_x := T|_{\mathcal{E}_x} : \mathcal{E}_x \rightarrow \mathcal{F}_x\) is an adjointable operator between Hilbert \(\mathcal{A}_x\)-modules.
- \(\sup_{x \in X} \| T_x \|_{\mathcal{E}_x \rightarrow \mathcal{F}_x} < \infty\).

If \(\mathcal{E}\) and \(\mathcal{F}\) are Hilbert \(\Gamma_0(\mathcal{A}, p)\)-modules, and if \(T : \mathcal{E} \rightarrow \mathcal{F}\) is an adjointable operator, then it is not difficult to show that the \(\Gamma_0(\mathcal{A}, p)\)-linearity of \(T\) give us a morphism \(T\) between the \((\mathcal{A}, p)\)-bundles of Hilbert \(C^*\)-modules corresponding to \(\mathcal{E}\) and \(\mathcal{F}\).

This observation leads us the following categorical result.

**Proposition (H., 2017)**

There is a natural isomorphism \(\tilde{\mathcal{S}}\) from the category of Hilbert \(\Gamma_0(\mathcal{A}, p)\)-modules (with morphisms given by the usual adjointable operators) to the category of \((\mathcal{A}, p)\)-bundles of Hilbert \(C^*\)-modules over \(X\) (with morphisms as described above).
Definition (Groupoids)

A groupoid is a small category $\mathcal{G}$ in which every morphism is an isomorphism. The set of objects of $\mathcal{G}$ is called the unit space of $\mathcal{G}$ and is denoted by $\mathcal{G}^{(0)}$. One also has maps $s, t : \mathcal{G} \rightarrow \mathcal{G}^{(0)}$ that take each morphism to its source and target respectively.

Definition (Groupoid Dynamical Systems)

A groupoid dynamical system is a quadruple $(\mathcal{G}, \alpha, \mathcal{A}, p)$ with the following properties:

- $\mathcal{G}$ is a locally compact Hausdorff groupoid (this means that $\mathcal{G}^{(0)}$ is Hausdorff, the groupoid operations (including $s$ and $t$) are continuous, and $s$ and $t$ are open).
- $(\mathcal{A}, p)$ is a $C^*$-bundle over $\mathcal{G}^{(0)}$.
- $\alpha$ is a $\mathcal{G}$-indexed family of $C^*$-algebraic isomorphisms governed by the following:
  - $\alpha_\gamma : \mathcal{A}_{s(\gamma)} \rightarrow \mathcal{A}_{t(\gamma)}$ for each $\gamma \in \mathcal{G}$.
  - $\alpha_{\gamma_1} \circ \alpha_{\gamma_2} = \alpha_{\gamma_1 \gamma_2}$ for each $(\gamma_1, \gamma_2) \in \mathcal{G}^{(2)} := \mathcal{G} \times_s \mathcal{G}$.
  - The map $\left\{ \begin{array}{c} \mathcal{G} \times_p \mathcal{A} \\ (\gamma, a) \end{array} \rightarrow \mathcal{A} \alpha_\gamma(a) \right\}$ is continuous.

From now on, $(\mathcal{G}, \alpha, \mathcal{A}, p)$ denotes a groupoid dynamical system.
A Hilbert \((\mathcal{G}, \alpha, \mathcal{A}, p)\)-module is a pair \((\mathcal{E}, \Phi)\) with the following properties:

- \(\mathcal{E}\) is a Hilbert \(\Gamma_0(\mathcal{A}, p)\)-module (where we let \((\mathcal{E}, q) := \mathcal{F}(\mathcal{E}))\).
- \(\Phi\) is a groupoid homomorphism from \(\mathcal{G}\) to
  
  \[
  \text{Iso}(\mathcal{E}, q) := \{(t(\gamma), T, s(\gamma)) \mid \gamma \in \mathcal{G} \text{ and } T \in \text{Isom}(\mathcal{E}_{s(\gamma)}, \mathcal{E}_{t(\gamma)})\}
  \]

  such that for each \(\gamma \in \mathcal{G}\), there is a linear isometry \(\Phi_\gamma \in \text{Isom}(\mathcal{E}_{s(\gamma)}, \mathcal{E}_{t(\gamma)})\) giving
  
  \[
  \Phi(\gamma) = (t(\gamma), \Phi_\gamma, s(\gamma)).
  \]

- The map \(\mathcal{G} \times_q \mathcal{E} \to \mathcal{E} \quad (\gamma, \zeta) \mapsto \Phi_\gamma(\zeta)\) is continuous.
- \(\Phi(x) = (x, \text{Id}_{\mathcal{E}_x}, x)\) for each \(x \in \mathcal{G}^{(0)}\).
- \(\Phi_\gamma(\zeta \bullet_{s(\gamma)} a) = \Phi_\gamma(\zeta) \bullet_{t(\gamma)} \alpha_\gamma(a)\) for each \(\gamma \in \mathcal{G}\), \(\zeta \in \mathcal{E}_{s(\gamma)}\) and \(a \in \mathcal{A}_{s(\gamma)}\).
- \(\langle \Phi_\gamma(\zeta), \Phi_\gamma(\eta) \rangle_{\mathcal{E}_{t(\gamma)}} = \alpha_\gamma \left(\langle \zeta, \eta \rangle_{\mathcal{E}_{s(\gamma)}}\right)\) for each \(\gamma \in \mathcal{G}\) and \(\zeta, \eta \in \mathcal{E}_{s(\gamma)}\).
The Category of Hilbert \((\mathcal{G}, \alpha, \mathcal{A}, p)\)-Modules

The class of Hilbert \((\mathcal{G}, \alpha, \mathcal{A}, p)\)-modules forms a category.

Given two objects \((\mathcal{E}, \Phi)\) and \((\mathcal{F}, \Psi)\) of this class, a morphism from \((\mathcal{E}, \Phi)\) to \((\mathcal{F}, \Psi)\) is an adjointable operator \(T : \mathcal{E} \to \mathcal{F}\) that satisfies the following equivariance property: If \(\mathcal{I} := \mathfrak{F}(T)\), then \(\mathcal{I}_{t(\gamma)} \circ \Phi_\gamma = \Psi_\gamma \circ \mathcal{I}_{s(\gamma)}\) for each \(\gamma \in \mathcal{G}\).

The equivariance property means that if \((\mathcal{E}, q) := \mathfrak{F}(\mathcal{E})\) and \((\mathcal{F}, r) := \mathfrak{F}(\mathcal{F})\), then the following diagram commutes:

\[
\begin{array}{ccc}
\mathcal{E}_{s(\gamma)} & \xrightarrow{\mathcal{I}_{s(\gamma)}} & \mathcal{F}_{s(\gamma)} \\
\Phi_\gamma \downarrow & & \downarrow \Psi_\gamma \\
\mathcal{E}_{t(\gamma)} & \xrightarrow{\mathcal{I}_{t(\gamma)}} & \mathcal{F}_{t(\gamma)}
\end{array}
\]
Examples of Hilbert \((\mathcal{G}, \alpha, \mathcal{A}, p)\)-Modules

**Example (An elementary example)**

A simple non-trivial example of a Hilbert \((\mathcal{G}, \alpha, \mathcal{A}, p)\)-module is \((\Gamma_0(\mathcal{A}, p), \alpha)\), where we make use of the isomorphism \(\Gamma_0(\mathcal{A}, p)/I_x \cong \mathcal{A}_x\) for all \(x \in \mathcal{G}^{(0)}\).

We now suppose that the groupoid \(\mathcal{G}\) has a left Haar system \((\lambda^x)_{x \in \mathcal{G}^{(0)}}\) of measures.

**Example (The standard Hilbert \((\mathcal{G}, \alpha, \mathcal{A}, p)\)-module (H., 2017))**

Let \((t^* \mathcal{A}, t^* p)\) denote the \(C^*\)-bundle over \(\mathcal{G}\) formed by pulling back the bundle \((\mathcal{A}, p)\) over \(\mathcal{G}^{(0)}\) under \(t\). Complete \(\Gamma_c(t^* \mathcal{A}, t^* p)\) to a Hilbert \(\Gamma_0(\mathcal{A}, p)\)-module \(\Gamma^2(t^* \mathcal{A}, t^* p)\).

The corresponding \((\mathcal{A}, p)\)-bundle of Hilbert \(C^*\)-modules has the property that its fiber over each \(x \in \mathcal{G}^{(0)}\) is unitarily isomorphic to the Hilbert \(\mathcal{A}_x\)-module \(L^2(\mathcal{G}^x, \mathcal{A}_x)\).

There is a groupoid homomorphism \(\Theta : \mathcal{G} \to \text{Iso}(\mathcal{F}(\Gamma^2(t^* \mathcal{A}, t^* p)))\) such that

\[
\forall \phi \in C_c \left(\mathcal{G}^s(\gamma), \mathcal{A}_s(\gamma)\right) : \quad \Theta_{\gamma}(\iota_{s(\gamma)}(\phi)) = \iota_{t(\gamma)} \left( \left\{ G^{t(\gamma)} \rightarrow \mathcal{A}_{t(\gamma)} \right\} \right),
\]

where \(\iota_x\) denotes the canonical dense linear embedding of \(C_c(\mathcal{G}^x, \mathcal{A}_x)\) into \(L^2(\mathcal{G}^x, \mathcal{A}_x)\). The pair \((\Gamma^2(t^* \mathcal{A}, t^* p), \Theta)\) is called the standard \((\mathcal{G}, \alpha, \mathcal{A}, p)\)-module.
Let \((\mathcal{E}, \Phi)\) be a Hilbert \((\mathcal{G}, \alpha, \mathcal{A}, p)\)-module with \((\mathcal{E}, q) := \mathcal{F}(\mathcal{E})\). For any \(\zeta \in \mathcal{E}\), define two linear operators

\[
\langle \zeta \rangle : \Gamma_0(\mathcal{E}, q) \to \Gamma_b(t^*\mathcal{A}, t^*p) \quad \text{and} \quad |\zeta\rangle : \Gamma_c(t^*\mathcal{A}, t^*p) \to \Gamma_0(\mathcal{E}, q),
\]
called the bra and ket of \(\zeta\) respectively, by

\[
\forall \sigma \in \Gamma_0(\mathcal{E}, q) : \quad \langle \zeta \rangle(\sigma) := \left\{ \begin{array}{l} \mathcal{G} \to t^*\mathcal{A} \\ \gamma \mapsto \langle \Phi_\gamma(\zeta + \mathcal{E}s(\zeta)), \sigma(t(\gamma)) \rangle_{\mathcal{E}t(\gamma)} \end{array} \right\};
\]

\[
\forall \phi \in \Gamma_c(t^*\mathcal{A}, t^*p) : \quad |\zeta\rangle(\phi) := \left\{ \begin{array}{l} \mathcal{E} \to \mathcal{E} \\ X \mapsto \int_\mathcal{G} \Phi_\gamma(\zeta + \mathcal{E}x) \bullet x \phi(\gamma) \, d\lambda^X(\gamma) \end{array} \right\}.
\]

Given any \(\zeta, \eta \in \mathcal{E}\), we have \(\langle \zeta | \eta \rangle[\Gamma_c(t^*\mathcal{A}, t^*p)] \subseteq \Gamma_b(t^*\mathcal{A}, t^*p)\).
Let \((\mathcal{E}, \Phi)\) be a Hilbert \((\mathcal{G}, \alpha, \mathcal{A}, p)\)-module with \((\mathcal{E}, q) := \mathcal{F}(\mathcal{E})\). We then call \(\zeta \in \mathcal{E}\) a square-integrable vector if and only if for any section \(\sigma \in \Gamma_0(\mathcal{E}, q)\) and any net \((\varphi_i)_{i \in I}\) of functions in \(C_c(\mathcal{G}, [0, 1])\) that converges uniformly to 1 on all compact subsets of \(\mathcal{G}\), the corresponding net \((\iota(\varphi_i \cdot \langle \zeta | \sigma \rangle))_{i \in I}\) in \(\Gamma^2(t^*A, t^*p)\) is also convergent. The set of square-integrable vectors is denoted by \((\mathcal{E}, \Phi)_{si}\).

We call \((\mathcal{E}, \Phi)\) a square-integrable representation of \((\mathcal{G}, \alpha, \mathcal{A}, p)\) if and only if \((\mathcal{E}, \Phi)_{si}\) is dense in \(\mathcal{E}\).

Using the concept of square-integrability, one can, in principle, achieve the following:

- A generalization of J. Brown’s results on proper groupoid actions on C*-algebras.
- The construction of generalized fixed-point algebras for proper groupoid actions on Hilbert C*-modules.
- A classification of reduced groupoid crossed products.
- An equivariant version of Kasparov’s Stabilization Theorem for groupoids.
- A definition of equivariant KK-theory for groupoid actions.

One only has to mimic the proofs given in [Meyer, 2001, Meyer, 2001] and [H, 2016].
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