Twists over étale groupoids and twisted vector bundles

Carla Farsi and Elizabeth Gillaspy

University of Colorado - Boulder and WWU - Münster

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Motivation

For a groupoid $\mathcal{G}$ with $\mathcal{G}^{(0)} = M$ and a twist $\mathcal{R}$ over $\mathcal{G}$,

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  [TXLG04] conjectured that their existence is implied by $[\mathcal{R}]$ torsion.

When do twisted vector bundles exist?

Theorem (Farsi-G, [FG16])

Let $\mathcal{G}$ be an étale groupoid and let $\mathcal{R}$ be a twist over $\mathcal{G}$, of order n in $H^2(\mathcal{G}, \mathcal{S})$. Suppose that the classifying space $B\mathcal{G}$ is a compact CW complex, and that the principal $PU(n)$-bundle over $\mathcal{G}^{(0)}$ induced by $\mathcal{R}$ lifts to a $U(n)$ principal bundle. Then $\mathcal{R}$ admits a twisted vector bundle.

Outline

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- Proof of main Theorem

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- A vector bundle is NOT étale.

Twists over groupoids

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$$r(\gamma) = r(p(\gamma)), \quad s(\gamma) = s(p(\gamma)), \quad p(\gamma\eta) = p(\gamma)p(\eta)\, \forall\, (\gamma, \eta) \in \mathcal{R}^{(2)}$$

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Example

If $c : \mathcal{G}^{(2)} \to \mathbb{T}$ is a continuous 2-cocycle, then $\mathcal{G} \times \mathbb{T}$ is a twist over $\mathcal{G}$:

$$(g, z)(h, w) := (gh, c(g, h)zw).$$

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Product of twists is given by Baer sum:

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Alternatively, equivalence classes of twists correspond to S^1-gerbes over the stack $\mathfrak{X}_{\mathcal{G}}$.

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If $\mathcal{G}$ is étale, each fiber $\mathcal{G}^u$ is discrete, so we take λ^u to be counting measure.

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If $\mathcal{R}$ arises from a 2-cocycle c, then $C_r^*(\mathcal{G}; \mathcal{R}) \cong \overline{C_c(\mathcal{G}, c)}$:

$$f *_c g(\gamma) = \int_{\mathcal{G}} f(\gamma\eta)g(\eta^{-1})c(\gamma\eta, \eta^{-1})\, d\lambda^{s(\gamma)}(\eta).$$

Twisted vector bundles

Let $\mathcal{R}$ be a twist over $\mathcal{G}$ (principal $\mathbb{T}$-bundle over $\mathcal{G}$). A twisted vector bundle is a vector bundle

$$\pi : E \to \left(\mathcal{G}^{(0)} = \mathcal{R}^{(0)}\right)$$

which admits an action of $\mathcal{R}$ such that, for all $z \in \mathbb{T},\ \gamma \in \mathcal{R},\ e \in E$ with $\pi(e) = s(\gamma)$,

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Proposition (TXLG)

If $(\mathcal{G}, \mathcal{R})$ admits a twisted vector bundle of rank n, then $\mathcal{R}$ represents a class of order n in $H^2(\mathcal{G}, \mathcal{S})$.

Proof:

Classifying space

One way to realize the standard k-simplex Δ_k:

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$$((g_1, \ldots, g_k),(t_1, \ldots, t_i, t_i, t_{i+1}, \ldots, t_{k-1})) \\ \sim ((g_1, \ldots, g_i g_{i+1}, \ldots, g_k), (t_1, \ldots, t_i, t_{i+1}, \ldots, t_{k-1})).$$

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Think of groupoid k-tuples as labeling k-simplices: the various possible partial products label the faces.

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Note:

- For all k, the map $\phi_k : \mathcal{G}^{(k)} \to B\mathcal{G}$ given by

$$\phi_k(g_1, \ldots, g_k) = [(g_1, \ldots, g_k), (0, \ldots, 0)]$$

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$$[((g_1, \ldots, g_k), m), (t_1, \ldots, t_k)] \mapsto [(m, (g_1, \ldots, g_k), (t_1, \ldots, t_k))].$$

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When do these hypotheses hold?

An example

$$\mathbb{R}P^2 = \{(\rho,\theta) : 0 \leq \rho \leq 1, 0 \leq \theta < 2\pi\}/\sim \text{ where } (1,\theta) \sim (1,\theta+\pi).$$

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Fix $x \in \mathbb{R}\backslash\mathbb{Q}$. Set $\mathcal{G} := M \rtimes_\alpha \mathbb{Z}$, where

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Künneth Theorem calculations tell us
$\mathbb{Z}/2\mathbb{Z} \subseteq Tw(\mathcal{G}) \cong H^2(M, \mathbb{Z})$;

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$\mathbb{R}P^2 = \{(\rho, \theta) : 0 \leq \rho \leq 1, 0 \leq \theta < 2\pi\}/\sim$ where $(1, \theta) \sim (1, \theta + \pi)$.

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$$M = \mathbb{R}P^2 \times S^4; \quad \alpha([\rho, \theta], z) = ([\rho, \theta + \rho x], z).$$

Since M is compact, $B\mathcal{G} \cong M \times_{\mathbb{Z}} \mathbb{R}$ is too.

Künneth Theorem calculations tell us
$\mathbb{Z}/2\mathbb{Z} \subseteq Tw(\mathcal{G}) \cong H^2(M, \mathbb{Z})$; moreover, for any n, the obstruction to lifting a $PU(n)$-bundle over M to a $U(n)$-bundle lives in

$$H^2(M, \mathbb{T}) \cong H^3(M, \mathbb{Z}) \cong H^3(\mathbb{R}P^2, \mathbb{Z}) \otimes H^0(S^4, \mathbb{Z}) = 0.$$

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