Cuntz-Pimsner algebras from groupoid actions

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Outline

- We recall the concept of a groupoid action on spaces, on graphs, on C^*-algebras, and on C^*-correspondences.
- Groups act on objects, but groupoids act on fibered objects.
- We consider several C^*-correspondences constructed from groupoid actions and their associated Cuntz-Pimsner algebras.
- There are connections with crossed products, graph C^*-algebras, Doplicher-Roberts algebras and with algebras of self-similar actions.
- We illustrate with some examples.

Groupoid actions on spaces

- In this talk, G denotes a locally compact Hausdorff groupoid with unit space G^0 and range and source maps $r, s : G \to G^0$.
- We say that G acts on a locally compact Hausdorff space X if there is a continuous, open surjection $\pi : X \to G^0$ and a continuous map

$$G * X \to X, \quad \text{write} \quad (g, x) \mapsto g \cdot x,$$

where $G * X = \{(g, x) \in G \times X \mid s(g) = \pi(x)\}$, such that

$$\pi(g \cdot x) = r(g), \;\; g_2 \cdot (g_1 \cdot x) = (g_2 g_1) \cdot x, \;\; \pi(x) \cdot x = x.$$

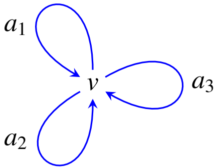
- For example, G acts on its unit space G^0 by $g \cdot s(g) = r(g)$. If there is only one orbit, the groupoid is transitive.
- A transitive groupoid is of the form $G^0 \times H \times G^0$ where H is the isotropy group.

Groupoid actions on graphs

- Let $E = (E^0, E^1, r, s)$ be a discrete graph.
- We say that the groupoid G with $G^0 = E^0$ acts on E if there is a map $G * E^1 \to E^1$ such that

$$s(g \cdot e) = g \cdot s(e), \quad r(g \cdot e) = g \cdot r(e),$$

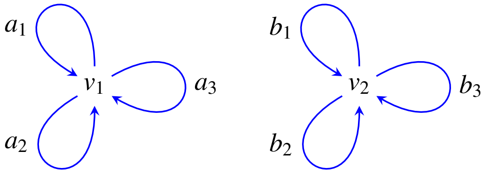
  where $G * E^1 = \{(g, e) : s(g) = r(e)\}$
- **Example**. Let E be the graph



- The symmetric group S_3 acts on E by $g \cdot a_i = a_{g(i)}, \;\; g \in S_3$ and determines an action on $C^*(E) = \mathcal{O}_3$.
- The crossed product $\mathcal{O}_3 \rtimes S_3$ is SME to a graph algebra and to the Doplicher-Roberts algebra of the permutation representation of S_3.

Example

- Let E be the graph



- The transitive groupoid G with unit space $E^0 = \{v_1, v_2\}$ and isotropy S_3 acts on E by permutations.
- The action is compatible with the Cuntz relations, so we get an action of G on $\mathcal{O}_3 \oplus \mathcal{O}_3$.
- What is $(\mathcal{O}_3 \oplus \mathcal{O}_3) \rtimes G$?

Groupoid actions on C^*-algebras

- Recall that a $C_0(X)-$algebra is a C^*-algebra A together with a homomorphism $\theta : C_0(X) \to ZM(A)$ such that $\overline{\theta(C_0(X))A} = A$.
- For each $x \in X$ we can define the fiber A_x as $A/\overline{\theta(I_x)A}$ where

$$I_x = \{f \in C_0(X) : f(x) = 0\}.$$

- A $C_0(X)$-algebra A gives rise to an upper semicontinuous C^*-bundle $\mathcal{A}$ such that $A \cong \Gamma_0(X, \mathcal{A})$.
- We say that G acts on a $C_0(G^0)$-algebra A if for each $g \in G$ there is an isomorphism $\alpha_g : A_{s(g)} \to A_{r(g)}$ such that

$$\alpha_{g_1 g_2} = \alpha_{g_1} \circ \alpha_{g_2} \text{ for } (g_1, g_2) \in G^{(2)}.$$

- We also write $g \cdot a$ for $\alpha_g(a)$.

Groupoid actions on C^*-correspondences

- Let $\mathcal{H}$ be a Hilbert module over a $C_0(X)$-algebra A. Define the fibers $\mathcal{H}_x := \mathcal{H} \otimes_A A_x$ which are Hilbert A_x-modules.
- The set $\mathrm{Iso}(\mathcal{H})$ of $\mathbb{C}$-linear isomorphisms between fibers becomes a groupoid with unit space X.
- Assume moreover that $\mathcal{H}_x$ is a C^*-correspondence over A_x and that G with $G^0 = X$ acts on A.
- An action of G on $\mathcal{H}$ is given by a homomorphism $\rho : G \to \mathrm{Iso}(\mathcal{H})$ where $\rho(g) : \mathcal{H}_{s(g)} \to \mathcal{H}_{r(g)}$ with $\rho(g) = I$ if $g \in G^0$, such that

$$\langle \rho(g)\xi, \rho(g)\eta \rangle_{r(g)} = g \cdot \langle \xi, \eta \rangle_{s(g)},$$

for $\xi, \eta \in \mathcal{H}_{s(g)}$ and

$$\rho(g)(\xi a) = (\rho(g)\xi)(g \cdot a), \ \ \rho(g)(a\xi) = (g \cdot a)(\rho(g)\xi)$$

for $\xi \in \mathcal{H}_{s(g)}, \ a \in A_{s(g)}$.

Crossed products of C^*-correspondences

- **Theorem**. Let G be a groupoid with a Haar system acting on a C^*-correspondence $\mathcal{H}$ over the $C_0(G^0)$-algebra A as above. Then the Katsura ideal $J_{\mathcal{H}}$ is G-invariant and we get an action of G on $\mathcal{O}_{\mathcal{H}}$, which becomes a $C_0(G^0)$-algebra.
- **Question**. Is $\mathcal{O}_{\mathcal{H}} \rtimes G \cong \mathcal{O}_{\mathcal{H} \rtimes G}$?
- Recall that the crossed product $\mathcal{H} \rtimes G = \mathcal{H} \otimes_A (A \rtimes G)$ becomes a C^*-correspondence over $A \rtimes G$ using the operations

$$\langle \xi, \eta \rangle(h) = \int g^{-1} \cdot \langle \xi(g), \eta(gh) \rangle_{r(g)} d\lambda^{r(h)}(g),$$

$$(\xi \cdot f)(h) = \int \xi(g)(g \cdot f(g^{-1}h)) d\lambda^{r(h)}(g),$$

$$(f \cdot \xi)(h) = \int f(g) \cdot (g \cdot \xi(g^{-1}h)) d\lambda^{r(h)}(g),$$

where $f \in C_c(G, r^*A)$, $\xi \in C_c(G, r^*\mathcal{H})$, and $r^*\mathcal{H} = \mathcal{H} \otimes_A r^*A$.

Exel-Pardo C^*-correspondences

- Let G be a group acting on a graph $E = (E^0, E^1, r, s)$. Then G acts on $A = C_0(E^0)$ and on $\mathcal{H} = \mathcal{H}_E$ by

$$\alpha_g(f)(v) = f(g^{-1}v),\ \ f \in C_0(E^0),\ \ \gamma_g(\xi)(e) = \xi(g^{-1}e),\ \ \xi \in C_c(E^1).$$

- A cocycle is a map $\varphi : G \times E^1 \to G$ such that $\varphi(g, e) \cdot s(e) = g \cdot s(e)$ for all $(g, e) \in G \times E^1$. Particular case $\varphi(g, e) = g$.
- Define an action of G on the Hilbert module $\mathcal{H}_E \rtimes_\gamma G$ by

$$(V_g\xi)(e, h) = \xi(g^{-1}e, \varphi(g^{-1}, e)h),\ \ \xi \in C_c(E^1 \times G).$$

- Together with the left action of $C_0(E^0)$ on $\mathcal{H}_E \rtimes_\gamma G$ given by

$$(\pi(f)\xi)(e, h) = f(r(e))\xi(e, h),$$

  we get a covariant representation of $(C_0(E^0), G, \alpha)$.
- From (π, V) we get a map $C_0(E^0) \rtimes_\alpha G \to \mathcal{L}(\mathcal{H}_E \rtimes_\gamma G)$, so $\mathcal{H}_E \rtimes_\gamma G$ becomes a C^*-correspondence $\mathcal{H}_E(\varphi)$ over $C_0(E^0) \rtimes_\alpha G$.
- The Cuntz-Pimsner algebra $\mathcal{O}_{\mathcal{H}_E(\varphi)}$ is the Exel-Pardo algebra associated to (E, G, φ).

Example: Katsura algebras

- To realize all Kirchberg algebras as topological graph algebras and to lift automorphisms of K-theory groups, Katsura introduced the algebras $\mathcal{O}_{A,B}$ for certain $n \times n$ matrices A, B with integer entries.
- Using generators and relations, Exel and Pardo proved that $\mathcal{O}_{A,B} \cong \mathcal{O}_{\mathcal{H}_E(\varphi)}$, where
- The matrix A is the incidence matrix of a finite graph E and the matrix B defines an action of $\mathbb{Z}$ on E.
- This action fixes the vertices and if the edges from i to j are labeled $e_{i,j,n}$ for $0 \leq n < A_{i,j}$, then $m \cdot e_{i,j,n} = e_{i,j,r}$, where $mB_{i,j} + n = qA_{i,j} + r$ with $0 \leq r < A_{i,j}$.
- The cocycle is $\varphi : \mathbb{Z} \times E^1 \to \mathbb{Z}, \ \ \varphi(m, e_{i,j,n}) = q$.

C^*-correspondences from self-similar actions

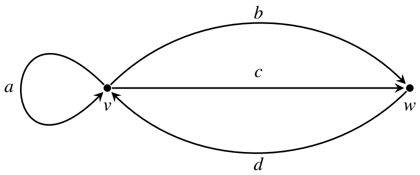
- Suppose E is a graph with no sources and define $E^* = \bigsqcup_{k\geq 0} E^k$, where E^k is the set of paths of length k.
- Denote by $T_E = \bigsqcup_{v\in E^0} vE^*$ the union of rooted trees (forest) with $T^0 = E^*, \quad T^1 = \{(\mu, \mu e) : \mu \in E^*, e \in E^1, s(\mu) = r(e)\}$.
- The set $\mathrm{Iso}(E^*)$ of partial isomorphisms $vE^* \to wE^*$ for $v, w \in E^0$ becomes a groupoid.
- An action of a groupoid G with unit space E^0 on E^* is given by a homomorphism $G \to \mathrm{Iso}(E^*)$.
- The action is *self-similar* if for every $g \in G$ and $e \in s(g)E^1$ there exists a unique $h \in G$ such that

$$g \cdot (e\mu) = (g \cdot e)(h \cdot \mu) \;\text{ for all }\; \mu \in s(e)E^*.$$

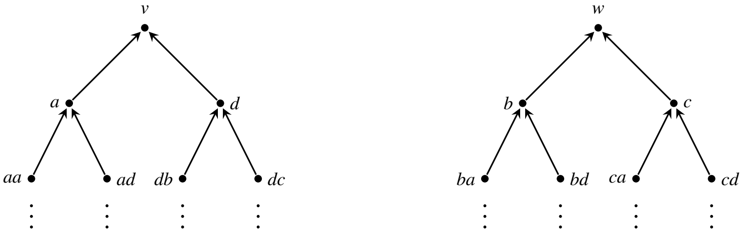
- We write $h = g|_e$, and the cocycle is given by $\varphi(g, e) = h$.

Example

Let E be the graph



with forest T_E



Example

- Define

$$g \cdot a = c, \;\; g|_a = v, \;\; g \cdot d = b, \;\; g|_d = h,$$
$$h \cdot b = a, \;\; h|_b = v, \;\; h \cdot c = d, \;\; h|_c = g$$

which determine an action of $G = \langle g, h \rangle$ on $T^0 = E^*$. For example,

$$g \cdot (aa) = ca, \;\; g \cdot (db) = ba, \;\; h \cdot (bd) = ad, \;\; h \cdot (ca) = dc.$$

- The space $\mathcal{M} = \mathcal{H}_E \otimes_{C(E^0)} C^*(G)$ becomes a Hilbert $C^*(G)$-module with usual inner product and right multiplication.
- The left multiplication is determined by the unitary representation

$$\rho : G \to \mathcal{L}(\mathcal{M}), \;\; \rho(g)(e \otimes f) = \begin{cases} (g \cdot e) \otimes \delta_{g|_e} f & \text{if } s(g) = r(e) \\ 0 & \text{otherwise,} \end{cases}$$

where $e \in E^1$, $f \in C^*(G)$ and $\delta_g \in C_c(G)$ denotes the point mass.
- Laca, Raeburn, Ramagge and Whittaker studied KMS states on the Toeplitz algebra $\mathcal{T}_{\mathcal{M}}$ and on the Cuntz-Pimsner algebra $\mathcal{O}_{\mathcal{M}}$.

The C^*-correspondence of a groupoid representation

- Let G be a groupoid with Haar system λ.
- Given a Hilbert bundle $\mathcal{H}$ over G^0 and a representation $\rho : G \to \mathrm{Iso}(\mathcal{H})$, define operations

$$\langle \xi, \eta \rangle(h) = \int \langle \xi(g^{-1}), \eta(g^{-1}h) \rangle_{s(g)} d\lambda^{r(h)}(g)$$

$$(\xi \cdot f)(h) = \int \xi(g) f(g^{-1}h) d\lambda^{r(h)}(g),$$

$$f \cdot \xi(h) = \int f(g) \rho(g) \xi(g) d\lambda^{r(h)}(g),$$

for $f \in C_c(G)$ and $\xi, \eta \in C_c(G, r^*\mathcal{H})$.

- The completion of $C_c(G, r^*\mathcal{H})$ becomes a C^*-correspondence $\mathcal{M}_\rho$ over $C^*(G)$, where the left action of $C_c(G)$ extends to a $*$-homomorphsim $\pi_\rho : C^*(G) \to \mathcal{L}(\mathcal{M}_\rho)$.

The compact group case

- **Theorem**. If G is a compact group and $\rho : G \to U(\mathcal{H})$ is a unitary representation with $\mathcal{H}$ separable, then $\mathcal{O}_{\mathcal{M}_\rho}$ is SME to a graph C^*-algebra.
  If $\pi_\rho : C^*(G) \to \mathcal{L}(\mathcal{H})$ is injective, then the graph has no sources.

- **Proof** (sketch). The C^*-algebra $C^*(G)$ decomposes as a direct sum of matrix algebras A_i with units p_i, indexed by the discrete set $\hat{G}$.
- Let E be the graph with vertex space $E^0 = \hat{G}$ and with edges determined by the A_j-A_i C^*-correspondences $p_j\mathcal{M}_\rho p_i$.
- $\mathcal{O}_{\mathcal{M}_\rho}$ is isomorphic to the C^*-algebra of a graph of C^*-correspondences in which we assign the algebra A_i at the vertex v_i and the minimal components of $p_j\mathcal{M}_\rho p_i$ for each edge joining v_i with v_j.
- By construction, this C^*-algebra is SME to $C^*(E)$.

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