Traces arising from regular inclusions

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Tracial states

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(2) how to be sure that our methods capture all possible tracial states on A.

In particular, we are interested in the case of *groupoid* and *graph* C^*-algebras.

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(1) an inclusion $B \subset A$ of a C^*-subalgebra (usually but not necessarily abelian) which contains an approximate identity for A;

(2) a conditional expectation $\mathbb{E} : A \to B$ (a completely positive linear bimodule map fixing B)

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For which tracial states $\phi \in T(B)$ is the extension $\phi \circ \mathbb{E}$ a *tracial state* on A? If S' is the set of such states, is the map $S' \to T(A)$ given by $\phi \mapsto \phi \circ \mathbb{E}$ a surjection?

Groupoid example

Example

Let G be an étale groupoid and let $A = C^*(G)$ be its C^*-algebra. The C^*-subalgebra $B = C_0(G^{(0)})$ is the range of a conditional expectation $\mathbb{E} : A \to B$ given by $f \mapsto f|_{G^{(0)}}$ for $f \in C_c(G)$. Any state ϕ on B is tracial and is represented by a measure μ.

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where μ is invariant if for every open set $U \subset G^{(0)}$ and open bisection $B \subset G$, we have $\mu(r(B) \cap U) = \mu(BUB^{-1})$. (Generalizes notion of invariant measure for a group action.)

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If $\tau \in T(A)$ is a tracial state, then $\phi = \tau|_B$ is a fully invariant (tracial) state on B.

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We found that under fairly mild assumptions this example can be reversed.

Normalization of conditional expectations

Definition

Let $\mathbb{E} : A \to B$ be a conditional expectation. We say that $\mathbb{E}$ is *normalized* by $n \in N(B)$ if $\mathbb{E}(nan^*) = n\mathbb{E}(a)n^*$ for all $a \in A$. (Similar for $N_0 \subset N(B)$.)

In the cases that we care about, the relevant conditional expectations will be normalized by a set of normalizers that generate A.

Invariant states, ctd.

Lemma

Suppose that $\phi \in T(B)$. Then $\phi \circ \mathbb{E}$ is centralized by n if and only if ϕ is n-invariant.

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Theorem (C., Nagy)

Suppose that $\mathbb{E} : A \to B$ is normalized by $\Sigma \subset N(B)$ and ϕ is a Σ-invariant tracial state on B. Then $\phi \circ \mathbb{E}$ is a tracial state when restricted to $C^*(\Sigma \cup B) \subset A$.

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If we take $A = C_r^*(G)$ and Σ to be the set of all "elementary normalizers", this gives a proof of the previous fact about tracial states on groupoid C^*-algebras.

Graph C^*-algebras

If $E = (E^0, E^1, r, s)$ is a directed graph, then there is a universal C^*-algebra $C^*(E)$ generated by a family $\{s_e, p_v\}_{e \in E^1, v \in E^0}$ such that

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For a directed path $\alpha = e_1 \dots e_n$, we denote the associated partial isometry $s_{e_1} \dots s_{e_n}$ by s_α.

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For a directed path $\alpha = e_1 \ldots e_n$, we denote the associated partial isometry $s_{e_1} \ldots s_{e_n}$ by s_α.Elements of the form $s_\alpha s_\beta^*$, for $\alpha, \beta \in E^*$ (finite path space), span the graph C^*-algebra.

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It is shown in [5] that there is a conditional expectation $\mathbb{E}$ from $C^*(E)$ onto $\mathcal{M}(E)$. So we obtain tracial states on $C^*(E)$ by extending states on $\mathcal{M}(E)$ via $\mathbb{E}$. In groupoid language, $\mathcal{M}(E)$ corresponds to $C^*(\text{Int Iso }(G_E)) \subset C^*(G_E) \cong C^*(E)$.

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Tomforde in [6] showed that the map $\tau \mapsto g_\tau$ is surjective onto the set of graph traces, using ordered K-theory.

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If we set H to be the set of all vertices as in the lemma, H is hereditary ($r(e) \in H$ implies $s(e) \in H$). Taking the saturation $\overline{H}$ one can check that $g(w) = 0$ for $w \in \overline{H}$ and g any graph trace.

We call the quotient graph $E \setminus \overline{H}$ the **tight subgraph** E_{tight}, and there is a canonical surjective $*$-homomorphism $\rho_{\text{tight}} : C^*(E) \to C^*(E \setminus \overline{H})$.

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We call the quotient graph $E \setminus \overline{H}$ the **tight subgraph** E_{tight}, and there is a canonical surjective $*$-homomorphism $\rho_{\text{tight}} : C^*(E) \to C^*(E \setminus \overline{H})$. Every tracial state on $C^*(E)$ factors through ρ, giving an isomorphism $T(C^*(E)) \cong T(C^*(E_{\text{tight}}))$.

Cyclically tagged graph traces

Definition

The *cyclic support* of a graph trace g is the set $\operatorname{supp}^c g$ of vertices v with $g(v) > 0$ that lie on cycles without entry. A *cyclically tagged graph trace* is a pair (g, μ), where g is a normalized graph trace and $\mu : \operatorname{supp}^c g \to \operatorname{Prob}(\mathbb{T})$. It is *consistent* if whenever v and w are on the same cycle, then $\mu(v) = \mu(w)$. The space of consistent cyclically tagged graph traces is denoted by $T_1^{\mathrm{CCT}}(E)$.

Example

If τ is a tracial state on $C^*(E)$, we obtain the graph trace g_τ as before, and the cyclic tagging $\mu = \mu_\tau$ is defined for $v \in \operatorname{supp}^c g$

$$\int_{\mathbb{T}} z^k d\mu_v = \frac{\tau(s_\lambda^k)}{\tau(p_v)} \qquad s(\lambda) = r(\lambda) = v \quad |\lambda| \text{ minimal.}$$

Invariant states and cyclically tagged graph traces

Theorem (C., Nagy)

If $(g, \mu) \in T_1^{\mathrm{CCT}}(E)$*, there is a state* $\phi_{(g,\mu)}$ *on* $\mathcal{M}(E)$ *which satisfies* $\phi_{(g,\mu)}(s_\alpha s_\alpha^*) = g(s(\alpha))$ *and* $\phi_{(g,\mu)}(s_\alpha s_\lambda^k s_\alpha^*) = g(s(\alpha)) \int_{\mathbb{T}} z^k \, d\mu_s(\alpha)$*.*

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Idea of proof

Divide the Gelfand spectrum Ω of $\mathcal{M}(E)$ into two parts. (One part will carry the graph trace and the other will carry the tagging.) Then define the state on $\mathcal{M}(E)$ by choosing a measure on Ω that is suitably invariant. $\square$

Parametrizing $T(C^*(E))$

Theorem (C., Nagy)

(1) *for any E, the map*

$$T_1^{\mathrm{CCT}}(E_{\mathrm{tight}}) \ni (g, \mu) \mapsto \tau_{(g,\mu)} \circ \rho_{\mathrm{tight}} \in T(C^*(E))$$

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(2) *if E is tight, then $\tau \mapsto (g_\tau, \mu_\tau)$ is an isomorphism from $T(C^*(E))$ onto $T_1^{\mathrm{CCT}}(E)$.*

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(2) *if E is tight, then $\tau \mapsto (g_\tau, \mu_\tau)$ is an isomorphism from $T(C^*(E))$ onto $T_1^{\mathrm{CCT}}(E)$.*

The key fact used in the proof is that $\mathcal{M}(E_{\mathrm{tight}}) \subset C^*(E_{\mathrm{tight}})$ has the extension property and so every tracial state on $C^*(E_{\mathrm{tight}})$ factors through the abelian core.

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Definition

Two (finite) paths λ and μ are *incomparable* if neither one contains the other as initial prefix. A vertex v is *essentially left infinite* if there is an infinite set $\{\lambda_k\}$ of finite paths that are pairwise incomparable and such that $s(\lambda_k) = v$ for all k.

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(2) Would like to understand how to obtain tracial states on general groupoid C^*-algebras in relation to isotropy groups ([4] already has something like this but not quite what we want)

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Thank you!