Diagonal preserving $*$-isomorphisms of Leavitt Path Algebras

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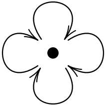
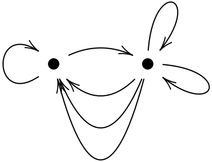
The Abrams-Tomforde Conjecture

Let $E = (E^0, E^1, r, s)$ be a directed graph.

$E =$ 

$F =$ 

- Construct a *Ring* $L(E)$ and a *C^*-algebra* $C^*(E)$.
- $L(E)$ is dense in $C^*(E)$.

Theorem (Abrams, Tomforde 2011)

If $L(E) \cong L(F)$ as $*$-algebras then $C^*(E) \cong C^*(F)$.

Conjecture (Abrams, Tomforde 2011)

If $L(E) \cong L(F)$ as rings then $C^*(E) \cong C^*(F)$.

Graph Groupoid

Let $E = (E^0, E^1, r, s)$ be a row-finite directed graph with no sources.

- A *path* $\mu = \mu_1 \cdots \mu_{|\mu|}$, $s(\mu_i) = r(\mu_{i+1})$.
  - The set of paths is E^*.
- An *infinite path* $x = x_1 x_2 \cdots$, $s(x_i) = r(x_{i+1})$.
  - The set of infinite paths is E^∞.
- $x \sim_k y$ if $x_{i+k} = y_i$ eventually.

$$G_E := \{(x, k, y) \in E^\infty \times \mathbb{Z} \times E^\infty : x \sim_k y\}.$$

  - Units: $G_E^{(0)} \leftrightarrow E^\infty \qquad (x, 0, x) \leftrightarrow x$.
  - $r(x, k, y) = x, \qquad s(x, k, y) = y$.
  - Multiplication: $(x, k, y)(y, \ell, z) = (x, k + \ell, z)$.
  - Inverse: $(x, k, y)^{-1} = (y, -k, x)$.
- Basis: $Z(\alpha, \beta) := \{(\alpha z, |\alpha| - |\beta|, \beta z)\} \quad \alpha, \beta \in E^*$.
  - $Z(\alpha, \alpha) \leftrightarrow Z(\alpha) := \{\alpha z\} \subset E^\infty$.
  - $Z(\alpha, \beta) \cdot \beta z \mapsto \alpha z$.

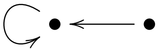
Topologically Principal

A groupid G is *topologically principal* if $\{x : r^{-1}(x) \cap s^{-1}(x) = \{x\}\}$ is dense in $G^{(0)}$. Recall

$$G_E := \{(x, k, y) \in E^\infty \times \mathbb{Z} \times E^\infty : x \sim_k y\}.$$

- Cycle: $\mu \in E^*$ with $r(\mu) = s(\mu)$.

- $(\mu\mu\cdots, k|\mu|, \mu\mu\cdots) \in G_E$.
  - The actions of $(\mu\mu\cdots, k|\mu|, \mu\mu\cdots)$ on E^∞ are indistinguishable for all k.
- A cycle has an entrance if there exists i with $r^{-1}(\mu_i) - \mu_i \neq \emptyset$.



- If μ has no entrance then $Z(\mu) = \{\mu\mu\cdots\}$: that is $\mu\mu\cdots$ is isolated.
- G_E is topologically principal if and only if every cycle has an entrance.

Leavitt path algebra

Let E be a directed graph and G_E the graph groupoid.

$$L(E) := \operatorname{span}_{\mathbb{C}}\{1_{Z(\alpha,\beta)} : \alpha, \beta \in E^*\}$$

- Grading:

$$L(E)_k := \operatorname{span}_{\mathbb{C}}\{1_{Z(\alpha,\beta)} : |\alpha| - |\beta| = k\}, \quad L(E) = \bigoplus_{\mathbb{Z}} L(E)_k.$$

- Multiplication: $f * g((x,k,y)) := \sum_{(x,\ell,z)} f(x,\ell,z)g(z,k-\ell,y)$.
  - $1_{Z(\alpha,\beta)} * 1_{Z(\mu,\nu)} = 1_{Z(\alpha,\beta)Z(\mu,\nu)} = 1_{Z(\alpha\delta,\nu\eta)}$ where $\beta\delta = \mu\eta$.
  - $1_{Z(\alpha)} * 1_{Z(\beta)} = 1_{Z(\alpha)\cap Z(\beta)} = 1_{Z(\beta)} * 1_{Z(\alpha)}$.
  - $D(E) := \operatorname{span}_{\mathbb{C}}\{1_{Z(\alpha)} : \alpha \in E^*\} \subset L(E)$ is commutative.
    - ★ $D(E) \subset L(E)_0$.
- Involution: $f^*(x,k,y) = \overline{f(y,-k,x)}$.
  - $1^*_{Z(\alpha,\beta)} = 1_{Z(\beta,\alpha)}$,
  - $f = \sum_{(\alpha,\beta)\in F} c_{\alpha,\beta} 1_{Z(\alpha,\beta)}$ then $f^* = \sum_{(\alpha,\beta)\in F} \overline{c_{(\alpha,\beta)}} 1_{Z(\beta,\alpha)}$.

C^*-algebra

Recall: $L(E) := \text{span}_{\mathbb{C}}\{1_{Z(\alpha,\beta)} : \alpha, \beta \in E^*\}$.

$$C^*(E) \quad \text{is the completion of} \quad L(E) \text{ in } \|\cdot\|.$$

- By definition $L(E)$ is dense in $C^*(E)$.
- Notice if E and F are graphs and $G_E \cong G_F$ then $L(E) \cong L(F)$ and $C^*(E) \cong C^*(F)$.

Idea: Use $L(E)$ and $L(F)$ to construct G_E and G_F so that if $L(E) \cong L(F)$ then so are G_E and G_F.

Normalizers

Let E be a directed graph and $L(E)$ the associated Leavitt path algebra.

Definition

$n \in L(E)$ is a normalizer *if $n^* D(E) n \cup n D(E) n^* \subset D(E)$. We denote the set of normalizers by $N(E)$.*

- $1_{Z(\alpha,\beta)} \in N(E)$.

$$1_{Z(\alpha,\beta)} 1_{Z(\mu)} 1_{Z(\beta,\alpha)} = 1_{Z(\alpha,\beta)} 1_{Z(\mu\eta,\alpha\delta)} = 1_{Z(\alpha\delta)}.$$

  where $\beta\delta = \mu\eta$.
  - So $N(E)$ contains a spanning set.

Note: Ara, Bosa, Hazrat and Sims use pairs (n, m) with $nD(E)m \cup mD(E)n \subset D(E)$. They insists $n \in L(E)_k$ for some k.

Strategy

Goal: Use $N(E)$ and $D(E)$ to define a groupoid W_E and show $W_E \cong G_E$. The typical strategy [Due to Alex Kumjian] is:

- Define a partial action of $N(E)$ on $\widehat{D(E)}$;
- Take W_E to be the groupoid of germs for this partial action.

Difficulties:

- The spectral theory used to define the partial action is not available.
- The groupoid of germs is topologically principal.
  - Idea is a modification of Brownlowe, Carlsen, Whittaker, 2015.

A partial Action

Take $n = \sum_{(\alpha,\beta)\in F} c_{(\alpha,\beta)} 1_{Z(\alpha,\beta)} \quad c_{(\alpha,\beta)} \neq 0.$

- $\mathrm{supp}(n) := \bigcup_{(\alpha,\beta)\in F} Z(\alpha,\beta)$.
- $\mathrm{Dom}(n) = \mathrm{supp}(n^*n) := \{x : n^*n(x) \neq 0\} \subset s(\mathrm{supp}(n))$.
- For $x \in \mathrm{Dom}(n)$, define

$$n \cdot x = r(\mathrm{supp}(n)x) := r(\{\gamma \in \mathrm{supp}(n) : s(\gamma) = x\}).$$

  - $|r(\mathrm{supp}(n)x)| = 1$ for all $x \in \mathrm{Dom}(n)$ [Proposition 3.3, B., Clark, an Huef.]
  - $|\mathrm{supp}(n)x|$ could be bigger than 1.
  - $n^*dn(x) = d(n \cdot x)n^*n(x)$.
  - If $(\alpha,\beta) \in F$ and $x \in Z(\beta) \cap \mathrm{Dom}(n)$, then

$$n \cdot x = 1_{Z(\alpha,\beta)} \cdot x = Z(\alpha,\beta) \cdot x.$$

Isolated points

Recall:

$$x \subset U \subset r(\mathrm{Iso(G)}) \Leftrightarrow x = \alpha\mu\mu\cdots \text{ with } \mu \text{ a cycle without entrance}$$
$$\Leftrightarrow Z(\alpha\mu) = \{x\}.$$

- G_E not topologically principal $\Leftrightarrow G_E^0 = E^\infty$ contains isolated points.

For x isolated define $p_x := 1_{Z(\alpha\mu)}$.

- Fact: for $n \in N(E)$ and $x = \alpha\mu\mu\cdots \in \mathrm{Dom}(n)$ isolated there exists k with $p_x n p_x \in L(E)_{k|\mu|}$. Lemma 3.8 [B., Clark, an Huef].
  - Brownlowe, et al. compute $K_0(p_x n p_x)$ here.

Generalized Weyl groupoid

Take $\mathcal{W}_E := \{(n, x) : n \in N(E), x \in \mathrm{Dom}(n)\}$.

Definition (B., Clark, an Huef)

$(n_1, x_1) \sim (n_2, x_2)$ if $x_1 = x_2 =: x$ and either

1. *x is isolated, $n_1 \cdot x = n_2 \cdot x$, and $p_x n_1^* n_2 p_x \in L(E)_0$, or*
2. *x not isolated and there exists open neighborhood of x, $V \subset Dom(n_1) \cap Dom(n_2)$ such that $n_1 \cdot y = n_2 \cdot y$ for all $y \in V$.*

- If $x = \alpha\mu\mu\cdots$ isolated $Z(\alpha\mu\mu, \alpha\mu)x = x = Z(\alpha\mu\mu, \alpha\mu\mu)x$, but
  - $n_1 = 1_{Z(\alpha\mu\mu,\alpha\mu)} \in L(E)_{|\mu|}$,
  - $n_2 = 1_{Z(\alpha\mu\mu,\alpha\mu\mu)} \in L(E)_0$,
  - $n_2^* n_1 \in L(E)_{|\mu|}$ so $(n_1, x) \not\sim (n_2, x)$.

Definition

- $W_E = \mathcal{W}_E/ \sim$,
  - $[n_1, n_2 \cdot x][n_2, x] = [n_1 n_2, x] \qquad [n_1, x]^{-1} = [n_1^*, n_1 \cdot x]$,
  - $W_E^0 := \{[d, x] : d \in D(E), x \in \mathrm{supp}(d)\}$.

Isomorphism of groupoids

Recall $W_E := \{[n, x] : x \in \mathrm{Dom}(n)\}$.

- W_E is a topological groupoid with basis

$$Z(n) := \{[n, x] : x \in \mathrm{Dom}(n)\}.$$

Theorem (B., Clark, an Huef)

The map

$$\Psi : G_E \to W_E \quad \textit{characterized by } (\mu x, |\mu| - |\nu|, \nu x) \quad \mapsto [1_{Z(\mu,\nu)}, \nu x]$$

is a well-defined isomorphism of topological groupoids.

- This theorem is an analogue of a C^*-result of Brownlowe, Carlsen, and Whittaker, 2015.
- Note: if $x = \delta y$, then $(\mu x, |\mu| - |\nu|, \nu x) = (\mu\delta y, |\mu\delta| - |\nu\delta|, \nu\delta y)$.
- Recall: If $n = \sum_{(\alpha,\beta)\in F} c_{(\alpha,\beta)} 1_{Z(\alpha,\beta)}$, $x \in \mathrm{Dom}(n) \cap Z(\beta)$ then

$$n \cdot x = 1_{Z(\alpha,\beta)} \cdot x.$$

Application to Abrams-Tomforde Conjecture

- G_E was constructed from the graph E and so *a priori* depends on the given graph.
- W_E was constructed using the subsets $N(E)$ and $D(E)$ of $L(E)$ so
  - W_E depends only on the inclusion of $D(E)$ in $L(E)$;
  - two Leavitt path algebras whose inclusions are the same will yield the same Generalized Weyl groupoid.

Theorem (B., Clark, an Huef)

Let E and F be directed graphs. Suppose

$$\begin{aligned}\Phi :& L(E) \to L(F) \\ \Phi|_{D(E)} :& D(E) \to D(F)\end{aligned}$$

are $*$-ring isomorphisms. Then

1. *$W_E \cong W_F$ as topological groupoids,*
2. *$G_E \cong G_F$ as topological groupoids,*
3. *$C^*(E) \cong C^*(F)$ as C^*-algebras.*

THANK YOU.