The Mackey Machine for Crossed Products

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GPOTS — 2009
Boulder Colorado

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Goals & Point of View

- Our main goal is to describe the ideal structure of crossed product C^*-algebras.
- Therefore a major objective is to describe the primitive ideal space together with its Jacobson topology.
- This is a difficult task and a general solution is well beyond reach at present.
- For example, even finding criteria for the simplicity of crossed products is extremely hard and only partial results are known.
- We will focus here on a small but important part of the process.
- First, we start with some basic definitions and examples.

Dynamical Systems

Definition

A dynamical system (A, G, α) consists of a separable C^*-algebra A, a second countable locally compact group G and a homomorphism $\alpha : G \to \operatorname{Aut} A$ such that $s \mapsto \alpha_s(a)$ is continuous for all $a \in A$.

Example (Transformation Groups)

Suppose that (G, X) is a locally compact transformation group. Then we can define a dynamical system by $\mathrm{lt} : G \to \operatorname{Aut} C_0(X)$ by $\mathrm{lt}_s(f)(x) := f(s^{-1} \cdot x)$. In fact, if $\alpha : G \to \operatorname{Aut} C_0(X)$ is a dynamical system, then $\alpha = \mathrm{lt}$ for an appropriate action of G on X.

Example (Single Automorphisms)

Suppose that $\phi \in \operatorname{Aut} A$. Then we can define $\alpha : \mathbf{Z} \to \operatorname{Aut} A$ by $\alpha_n(a) := \phi^n(a)$.

Definition

A covariant representation of (A, G, α) on a Hilbert space $\mathcal{H}$ is a pair (π, U) consisting of a representation $\pi : A \to B(\mathcal{H})$ and a unitary representation $U : G \to U(\mathcal{H})$ such that $\pi\big(\alpha_s(a)\big) = U_s \pi(a) U_s^*$ for all $a \in A$ and $s \in G$.

Example (Regular Representations)

Let (A, G, α) be any dynamical system and let $\rho : A \to B(\mathcal{V})$ be a representation. Then the associated regular representation on $L^2(G, \mathcal{V})$ is given by the covariant pair (π, U), where $\pi(a)h(r) = \rho\big(\alpha_r^{-1}(a)\big)\xi(r)$ and $U_s\xi(r) = \xi(s^{-1}r)$. Back

We can make $C_c(G, A)$ into a $*$-algebra with multiplication given by $f * g(s) := \int_G f(r)\alpha_r\big(g(r^{-1}s)\big)\, dr$, and involution given by $f^*(s) := \Delta(s^{-1})\alpha_s\big(f(s^{-1})^*\big)$. A covariant pair (π, U) defines a $*$-representation of $C_c(G, A)$:

$$\pi \rtimes U(f)h := \int_G \pi\big(f(r)\big) U_r h\, dr.$$

The universal norm on $C_c(G, A)$ is given by

$$\|f\| := \sup\{\, \|\pi \rtimes U(f)\| : (\pi, U) \text{ is convariant}\,\}.$$

The crossed product, $A \rtimes_\alpha G$, is the completion $\overline{(C_c(G, A), \|\cdot\|)}$. Every representation of $A \rtimes_\alpha G$ is the integrated form of a convariant representation. Thus the crossed product $A \rtimes_\alpha G$ can be thought of and a universal object for covariant representations of (A, G, α)

Suppose that (A, G, α) is a dynamical system and $\rho : A \to B(\mathcal{V})$ is a representation of A. Let $\operatorname{Ind}_e^G \rho := \pi \rtimes U$ be the integrated form of the associated regular representation on $L^2(G, \mathcal{V})$. Then if $f \in C_c(G, A)$,

$$(\operatorname{Ind}_e^G \rho)(f)\xi(r) = \int_G \rho\big(\alpha_r^{-1}(f(s))\big)\xi(s^{-1}r)\,ds.$$

Example (Generic Example)

If G is amenable and ρ is faithful, then it is an old result of Takai that $\operatorname{Ind}_e^G(\rho)$ is isometric for the universal norm. Then $A \rtimes_\alpha G$ is (isomorphic to) the closure in $B\big(L^2(G, \mathcal{V})\big)$ of $\{\, \big(\operatorname{Ind}_e^G \rho\big)(f) : f \in C_c(G, A)\,\}$.

To give some more concrete and nontrivial examples, we need a bit of machinery.

Morita Equivalence

- Note that we use simply "Morita equivalence" in place of what Rieffel called "strong Morita equivalence". Ordinary — or what we prefer to call "weak", or better yet "categorical" Morita equivalence — is of little use, and it seems peculiar to reserve the undecorated term for it.
- Formally, A and B are Morita equivalent if we can find an A–B-imprimitivity bimodule.

Definition

A Banach A–B-bimodule X is called an A–B-imprimitivity bimodule if it is both a left Hilbert A-module and a right Hilbert B-module such that

- The ideals ${}_A\langle X , X\rangle$ and $\langle X , X\rangle_B$ are dense in A and B, respectively.
- $\langle a \cdot x , y\rangle_B = \langle x , a^* \cdot y\rangle_B$ and ${}_A\langle x \cdot b , y\rangle = {}_A\langle x , y \cdot b^*\rangle$ and
- ${}_A\langle x , y\rangle \cdot z = x \cdot \langle y , z\rangle_B$.

- Of course, Morita equivalence is weaker than isomorphism.
- If A and B are separable — or even σ-unital — then the Brown-Green-Rieffel Theorem says that A and B are Morita equivalent if and only if A and B are stably isomorphic (i.e., $A \otimes \mathcal{K} \cong B \otimes \mathcal{K}$).
- Although much celebrated, the BGR-Theorem is probably not a good way to "understand" Morita theory.
- Instead, I encourage you to think in terms of representation theory.

Induced Representations

- Let X be a $A-B$-imprimitivity bimodule, and suppose that π is a representation of B on $\mathcal{H}_\pi$.
- We can form a Hilbert space $\mathsf{X} \otimes_B \mathcal{H}_\pi$ which is the completion of (the algebraic tensor product) $\mathsf{X} \odot \mathcal{H}_\pi$ with respect to the pre-inner product satisfying

$$(x \otimes \xi \mid y \otimes \eta) = \big(\pi\big(\langle y \,,\, x\rangle_{B}\big)\xi \mid \eta\big).$$

- We get an induced representation $\mathsf{X}\text{–}\mathrm{Ind}\,\pi$ of A on $\mathsf{X} \otimes_B \mathcal{H}_\pi$ via

$$\mathsf{X}\text{–}\mathrm{Ind}\,\pi(a)[x \otimes \xi] := [a \cdot x \otimes \xi].$$

- The map $\pi \mapsto \mathsf{X}\text{–}\mathrm{Ind}\,\pi$ establishes an equivalence between the categories of representations of A and representations of B.
- In particular, π is irreducible if and only if $\mathsf{X}\text{–}\mathrm{Ind}\,\pi$ is irreducible and every irreducible representation of A is of this form.

Mackey's Imprimitivity Theorem

Example (Mackey's Imprimitivity Theorem)

Suppose that H is a closed subgroup of G. In modern language, Mackey's Imprimitivity Theorem says that $C_0(G/H) \rtimes_{\mathrm{lt}} G$ is Morita equivalent to $C^*(H)$.

Example (Mackey's Induced Representations)

There is a natural map ϕ of $C^*(G)$ into $M\big(C_0(G/H) \rtimes_{\mathrm{lt}} G\big)$. Then we can realize the representation of G induced from a representation ω of H by $U_H^G\omega := \text{X–Ind}\,\omega \circ \phi$.

Remark (Stone-von Neumann Theorem)

The Imprimitivity Theorem implies that $C_0(G) \rtimes_{\mathrm{lt}} G$ is simple. Then it is not difficult to see that $C_0(G) \rtimes_{\mathrm{lt}} G \cong \mathcal{K}\big(L^2(G)\big)$.

More Examples

Example

1. With quite a bit more work, Green was able to use the Imprimitivity Theorem to show that $C_0(G/H) \rtimes_{\mathrm{lt}} G \cong C^*(H) \otimes \mathcal{K}\big(L^2(G/H)\big)$.
2. $C_0(\mathbf{R}/\mathbf{Z}) \rtimes_{\mathrm{lt}} \theta\mathbf{Z} \cong A_\theta$.
3. If $\alpha_s(a) = u_s a u_s^*$ for a strictly continuous homomorphism $u : G \to UM(A)$, then $A \rtimes_\alpha G \cong A \otimes_{\max} C^*(G)$.
4. Let $G_P = \{\, s \in G : P = s \cdot P \,\}$, where $s \cdot P := \alpha_s(P)$, be the stability group at P. If $sG_P \mapsto s \cdot P$ is a homeomorphism of G/G_P onto $\operatorname{Prim} A$, then $A \rtimes_\alpha G$ is Morita equivalent to $A/P \rtimes_{\alpha^P} G_P$.

Remark (Stability Groups)

The last example is meant to illustrate the maxim that stability groups play a key role in the ideal structure of crossed products.

If H is a subgroup of G, then $A \rtimes_{\alpha|_H} H$ is Morita equivalent to $E_H^G(A) := C_0(G/H, A) \rtimes_{\mathrm{lt}\otimes\alpha} G$ (via X_H^G) and there is a natural map $\phi : A \rtimes_\alpha G \to M\big(E_H^G(A)\big)$. We define the representation of $A \rtimes_\alpha G$ induced from the representation L of $A \rtimes_{\alpha|_H} H$ to be

$$\operatorname{Ind}_H^G L := \mathsf{X}_H^G\text{–}\operatorname{Ind}(L) \circ \phi.$$

Example (Regular Representations)

If we let $H = \{ e \}$ be the trivial subgroup and view a representation $\rho : A \to B(\mathcal{V})$ as a covariant representation of the degenerate system $(A, \{ e \}, \mathrm{id})$, then the corresponding induced representation is (equivalent to) the regular representation $\operatorname{Ind}_e^G \rho$ defined earlier.

Restriction and Induced Ideals

Proposition (Maps on Ideals)

There are continuous maps $\operatorname{Ind}_H^G : \mathscr{I}\big(A \rtimes_{\alpha|_H} H\big) \to \mathscr{I}\big(A \rtimes_\alpha G\big)$ *and* $\operatorname{Res} : \mathscr{I}\big(A \rtimes_\alpha G\big) \to \mathscr{I}(A)$ *such that*

$$\operatorname{Ind}_H^G(\ker L) = \ker\big(\operatorname{Ind}_H^G L\big) \quad \textit{and} \quad \operatorname{Res}\big(\ker(\pi \rtimes U)\big) = \ker \pi.$$

Remark (The Role of Res)

If L is a representation of $A \rtimes_\alpha G$ and if $I = \operatorname{Res} \ker L$, then L factors through $A/I \rtimes_{\alpha^I} G$.

In loose terms, the **Mackey Machine** (or the Green-Mackey-Rieffel Machine) for crossed products $A \rtimes_\alpha G$ is the process of recovering the primitive ideal space of $A \rtimes_\alpha G$ via induction from systems associated to the stability groups G_P for $P \in \operatorname{Prim} A$.

For motivation, let's look briefly at an example where everything works rather nicely. Namely, we'll look at transformation groups (G, X) with G abelian. Fix $x \in X$, $\omega \in \widehat{G}$ and let G_x be the stability group at x. Then evaluation at x, ev_x, is a representation of $C_0(X)$ and $(\mathrm{ev}_x, \omega|_{G_x})$ is a covariant representation of $\big(C_0(G), G_x, \mathrm{lt}\big)$. It is not hard to see that the induced representation $\operatorname{Ind}_{G_x}^{G}(\mathrm{ev}_x \rtimes \sigma|_{G_x})$ is equivalent to the representation $\operatorname{Ind}_{G_x}^{G}(\omega|_{G_x})$ on $B\big(L^2(G/G_x)\big)$ given by the covariant pair (M, U) where $M(\phi)h(\dot{s}) = \phi(s \cdot x)$ and $U_r h(\dot{s}) = \omega(r)h(r^{-1}\dot{s})$. Return

A Theorem from Antiquity

P1: Work of Mackey (1949) shows that $\operatorname{Ind}_{G_x}^G(\omega|_{G_x})$ is always an irreducible representation of $C_0(X) \rtimes_{\mathrm{lt}} G$.

P2: A deep theorem of Gootman, Rosenberg and Sauvageot (about which I'll have much more to say shortly) implies that if $K \in \operatorname{Prim}\bigl(C_0(X) \rtimes_{\mathrm{lt}} G\bigr)$ then there is $(x,\omega) \in X \times \widehat{G}$ such that $K = \ker\bigl(\operatorname{Ind}_{G_x}^G(\omega|_{G_x})\bigr)$.

Theorem (W 1981)

The map $(x,\omega) \mapsto \operatorname{Ind}_{G_x}^G(\omega|_{G_x})$ is a continuous open surjection of $X \times \widehat{G}$ onto $\operatorname{Prim}\bigl(C_0(X) \rtimes_{\mathrm{lt}} G\bigr)$. It induces a homeomorphism of $\operatorname{Prim}\bigl(C_0(X) \rtimes_{\mathrm{lt}} G\bigr)$ onto the quotient $(X \times \widehat{G})/\sim$, where $(x,\omega) \sim (y,\sigma)$ if and only if $\overline{G\cdot x} = \overline{G\cdot y}$ and $\sigma\bar{\omega} \in G_x^\perp = G_y^\perp$.

Moving Forward

The object of the exercise is to extend this result to more general crossed products. Ideally, this entails finding an easily described space Z (played by $X \times \widehat{G}$ above), and a surjection from Z to $\operatorname{Prim} A \rtimes_\alpha G$. Then we realize $\operatorname{Prim} A \rtimes_\alpha G$ as the quotient — perhaps even topologically. We need analogues of P1 to see that the map is into, and P2 to see that it is surjective.
In order to properly formulate such statements, we need the following.

Definition

A primitive ideal $K \in \operatorname{Prim}(A \rtimes_\alpha G)$ is induced if there is a $P \in \operatorname{Prim} A$ and a J in $\operatorname{Prim}(A \rtimes_{\alpha|_{G_P}} G_P)$ such that $\operatorname{Res} J = P$ and $\operatorname{Ind}_{G_P}^G J = K$.

To see that this is the "right" definition, let's look at a particularly nice case.

The Smooth Case = Regular Case = Nice Case

Theorem (The Regular Case)

Suppose that (A, G, α) is separable, that A is type I and that $G\backslash \operatorname{Prim} A$ is a T_0 topological space. Then every primitive ideal of $A \rtimes_\alpha G$ is induced. In fact, if R is an irreducible representation of $A \rtimes_\alpha G$, then there is a $P \in \operatorname{Prim} A$ and a irreducible representation $L = \pi \rtimes V$ of $A \rtimes_{\alpha|_{G_P}} G_P$ such that $\ker \pi = P$ and $R = \operatorname{Ind}_{G_P}^G L$.

Sketch of the Proof.

The hypotheses imply that each orbit $G \cdot P$ is locally closed in $\operatorname{Prim} A$ and homeomorphic to G/G_P; hence there is a subquotient $A(G \cdot P) = I/J$ of A whose primitive ideal space can be identified with $G \cdot P$. It follows that every irreducible representation of $A \rtimes_\alpha G$ is lifted from a subquotient of the form $A(G \cdot P) \rtimes_{\alpha^J} G$. Since $G \cdot P \cong G/G_P$, the latter is Morita equivalent to $A/P \rtimes_{\alpha^P} G_P$, and the result follows. □

The Effros-Hahn Conjecture

In 1967, Effros and Hahn conjectured that, provided G is amenable, every primitive ideal of a crossed product should be induced (even if the action of G on $\operatorname{Prim} A$ is pathological).

Many people worked on various versions of this conjecture, but the final qed was written by Gootman and Rosenberg in 1979.

Theorem (Gootman-Rosenberg-Sauvageot)

If (A, G, α) is separable and if G is amenable, then every primitive ideal of $A \rtimes_\alpha G$ is induced.

Example

Suppose that G is amenable, that G acts freely on $\operatorname{Prim} A$ and that A has no nontrivial G-invariant ideals. Then $A \rtimes_\alpha G$ is simple.

The Missing Piece

So why are we here?

The GRS-Theorem — and for that matter, our result for regular systems — does not address the following question.

Question

If $J \in \operatorname{Prim}(A \rtimes_{\alpha|_{G_P}} G_P)$ is such that $\operatorname{Res} J = P$, then must $\operatorname{Ind}_{G_P}^{G} J$ be primitive?

Lack of an answer for this question is unsatisfactory on several levels. In particular, it provides a real obstruction for a fine analysis of the ideal structure of crossed products where the ultimate goal of the Green-Mackey-Rieffel machine is to give a succinct description of $\operatorname{Prim}(A \rtimes_\alpha G)$ and its topology.

EHI and Strong-EHI

Definition

We say that (A, G, α) satisfies the Effros-Hahn Induction Property (EHI) if given $P \in \operatorname{Prim} A$ and a $J \in \operatorname{Prim}\big(A \rtimes_{\alpha|_{G_P}} G_P\big)$ such that $\operatorname{Res} J = P$, then $\operatorname{Ind}_{G_P}^{G} J$ is primitive. We say that (A, G, α) satisfies the strong Effros-Hahn Induction Property (strong-EHI) if given $P \in \operatorname{Prim} A$ and an irreducible representation $\rho \rtimes V$ of $A \rtimes_{\alpha|_{G_P}} G_P$ such that $\ker \rho = P$, then $\operatorname{Ind}_{G_P}^{G}(\rho \rtimes V)$ is irreducible.

Conjecture

Every separable dynamical system (A, G, α) satisfies EHI.

At least I Hope its True

Every separable dynamical system (A, G, α) satisfies EHI.

As we shall see, it is possible that every separable system satisfies strong-EHI.

What does the GRS-Theorem say about EHI?

Recall that a representation $\rho : A \to B(\mathcal{H})$ is called homogeneous if every nonzero subrepresentation of ρ has the same kernel as ρ.

$$\rho \in \hat{A} \implies \rho \text{ factorial} \implies \rho \text{ homogeneous} \implies \ker \rho \text{ prime.}$$

And prime ideals are primitive in separable C^*-algebras.

Proposition (Sauvageot)

Suppose that (A, G, α) is separable and that ρ is a homogeneous representation of A with kernel $P \in \operatorname{Prim} A$ and that $\rho \rtimes V$ is a homogeneous representation of $A \rtimes_{\alpha|_{G_P}} G_P$. Then $\operatorname{Ind}_{G_P}^{G}(\rho \rtimes V)$ is homogeneous.

The "extra" condition on ρ means that this proposition only directly implies that (A, G, α) satisfies EHI if we know that any irreducible representation $\rho \rtimes V$ of $A \rtimes_{\alpha|_{G_P}} G_P$ must have ρ homogeneous. Unfortunately, this is not always the case.

Upgrading to strong-EHI

Theorem (EW)

Suppose that (A, G, α) is separable and that $L = \rho \rtimes V$ is an irreducible representation of $A \rtimes_{\alpha|_{G_P}} G_P$ such that ρ homogeneous with kernel P. Then $\operatorname{Ind}_{G_P}^{G} L$ is irreducible.

Sketch of the Proof.

We can realize $\operatorname{Ind}_{G_P}^{G} L$ as a representation $\pi \rtimes U$ on $L^2(G/G_P, \mathcal{H})$ such that

$$\pi = \int_{G/G_P}^{\oplus} \pi_{\dot{s}} \, d\mu(\dot{s}).$$

Since ρ is homogeneous, each $\pi_{\dot{s}}$ is homogeneous with kernel $s \cdot P$. Then a deep result of Effros's forces

$$\pi(A)' \subset M\big(C_0(G/G_P)\big)',$$

where $M(\phi)h(\dot{s}) = \phi(\dot{s})h(\dot{s})$. Then any T which commutes with $\pi \rtimes U$ must also commute with $(M \otimes \pi) \rtimes U \cong \mathsf{X}_{G_P}^{G}$–Ind L. Since the latter is irreducible, T is a scalar. □

Our First Positive Result

Theorem

Suppose that (A, G, α) is separable and that points in Prim A *are locally closed.*[a] *Then (A, G, α) satisfies strong-EHI.*

[a] If A is of type I, then points in Prim $A \cong \hat{A}$ are always locally closed.

Sketch of the Proof.

When P is open in its closure, then we can prove that whenever $\rho \rtimes V$ is an irreducible representation of $A \rtimes_{\alpha|_{G_P}} G_P$, we must have ρ homogeneous. In that event, the previous theorem applies. □

Remark (Back to P1 & P2 in the "Antiquity Result")

The above result supplies another proof of Mackey's result (P1) in our motivational example. However, for A commutative, the irreducibility of the induced representation holds without any separability hypotheses.

Another Result in Support of the Conjecture

Theorem

Suppose that (A, G, α) is separable and that each stability group G_P is normal in G.[a] *Then (A, G, α) satisfies strong-EHI.*

[a] Note that this hypothesis holds automatically if G is abelian.

Sketch of the Proof.

Using twisted crossed products, we can write $A \rtimes_\alpha G$ as an iterated twisted crossed product $(A \rtimes_{\alpha|_{G_P}} G_P) \rtimes_\gamma^\tau G$. This allows us, in essence, to reduce to the case where the action is free. Then our theorem applies. □

What about the GRS-Theorem?

If G is amenable — or at least if all the G_P are amenable — then it seems reasonable to guess that the GRS-Theorem should provide an answer to our conjecture.

For example, suppose that G_P is amenable and $J \in \operatorname{Prim} A \rtimes_\alpha G_P$ with $\operatorname{Res} J = P$. The GRS-Theorem implies that $J = \operatorname{Ind}_{(G_P)_Q}^{G_P} K$ where $K \in \operatorname{Prim} A \rtimes_\alpha (G_P)_Q$ and $K = \ker(\sigma \rtimes W)$ with both $\sigma \rtimes W$ and σ homogeneous, and with $\ker \sigma = Q$. If $G_Q \subset G_P$, so that $(G_P)_Q = G_Q$, then

$$\operatorname{Ind}_{G_P}^{G} J = \operatorname{Ind}_{G_P}^{G} \operatorname{Ind}_{G_Q}^{G_P} K = \operatorname{Ind}_{G_Q}^{G} K,$$

which is primitive by Sauvageot's result. Unfortunately, we can't see that G_Q need be a subset of G_P in this special setting.

A Positive or a Negative Result: You Decide.

But what we can prove is this.

Proposition

Suppose that (A, G, α) is separable, that all the G_P are amenable and that whenever $P, Q \in \operatorname{Prim} A$ satisfy

$$P = \bigcap_{s \in G_P} s \cdot Q,$$

then either $G_Q \subset N(G_P)$ or $G_P \subset N(G_Q)$ (where $N(H) = \{\, s \in G : sHs^{-1} \subset H \,\}$ is the normalizer of H in G). Then (A, G, α) satisfies EHI.

If nothing else, this suggests that any counterexample to the conjecture will have an "interesting" orbit structure.

Open Questions & Future Directions

- It is still open as to whether every separable C^*-dynamical system satisfies EHI — or even strong-EHI.
- These questions are open even if G is amenable.
- Marius Ionescu and I are currently working on extending the theory to groupoid dynamical systems.
- For groupoid systems, Renault has proved a version of the GRS-Theorem. However, Renault's result does not directly address the issue of "inducing from stability groups".
- In fact, it is not even clear exactly what "inducing from stability groups" should mean in this generality.
- However, in the scalar case, we have some positive results.

For Example: The Scalar Case

- If G is a second countable, locally compact Hausdorff groupoid with a Haar system, then there is a notion of induction from a stability group $H := G(u) := \{ x \in G : s(x) = u = r(x) \}$ which is completely analogous to the situation in the case of (ordinary) dynamical systems described previously.
- Here, $G_u = s^{-1}(u)$ is a principal right H-space, and we can form the imprimitivity groupoid $H^G = G_u *_r G_u/H$.
- H^G is equivalent to H and Renault's Equivalence Theorem implies that $C^*(H^G)$ is Morita equivalent to $C^*(H)$. There is a homomorphism ϕ of $C^*(G)$ into $M(C^*(H^G))$ and we have $\operatorname{Ind}_H^G L := \text{X–Ind}\, L \circ \phi$.

Groupoid C^*-algebras satisfy strong-EHI

Theorem (Ionescu & W)

Suppose that G is a second countable, locally compact Hausdorff groupoid with a Haar system. Let $H = G(u)$ be a stability group. If L is an irreducible representation of H, then $\operatorname{Ind}_H^G L$ is an irreducible representation of G.

Sketch of the proof in a special case.

If the orbit $[u] := r\big(s^{-1}(u)\big) = r(G_u)$ is closed, then $C^*(G|_{[u]})$ is a quotient of $C^*(G)$. However, $G|_{[u]}$ is equivalent to $G(u)$. Therefore Renault's Equivalence Theorem implies $C^*(G|_{[u]})$ is Morita equivalent to $C^*(G(u))$, and the result follows in this special case. □

In general, $[u]$ can be badly embedded in G. Nevertheless, compact subsets of the orbit sit inside G sufficiently nicely to take advantage of the ideas in the above argument.

GRS For Groupoid C^*-algebras

Based on some very deep work of Renault — including a version of Effros's direct integral result for groupoid crossed products — Marius Ionescu and I were able to prove the following.

Theorem (Ionescu and W)

Suppose that G is an amenable locally compact second countable groupoid with a Haar system. Then every primitive ideal of $C^*(G)$ is induced. That is, if $K \in \operatorname{Prim} C^*(G)$, then there is a $u \in G^{(0)}$ and a $J \in \operatorname{Prim} C^*(G(u))$ such that $K = \operatorname{Ind}_{G(u)}^{G} J$.

Even though the proof of the above theorem involves groupoid dynamical systems in a nontrivial way, we are still quite far from even formulating a good theory for general groupoid crossed products. But we're working on it.