Orbifold groupoids, sectors, and wreath products

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Orbifolds and the Inertia Orbifold

Orbifolds

Orbifolds were originally introduced as generalizations of manifolds formed by taking local charts whose domains are subsets of $\mathbb{R}^n/G$ where G is a finite group.

As an example, if G is a finite group acting smoothly on an manifold M, then M/G is an orbifold.

Orbifolds of the form M/G are called **global quotient orbifolds.**

Example: The $\mathbb{Z}_k$-Teardrop

The $\mathbb{Z}_k$-teardrop Q is an example of an orbifold that is not a global quotient.

The Euler-Satake Characteristic

Definition

(Satake; 1957) Let $\mathcal{T}$ be a simplicial decomposition of a compact orbifold Q such that on the interior of any simplex σ, the isotropy group G_σ is constant. Then the **Euler-Satake characteristic of** Q is

$$\chi_{ES}(Q) = \sum_{\sigma \in \mathcal{T}} (-1)^{\dim \sigma} \frac{1}{|G_\sigma|}.$$

The Euler-Satake characteristic of the $\mathbb{Z}_k$-teardrop is $\frac{k+1}{k}$.

Example: The $\mathbb{Z}_k$-Teardrop

The $\mathbb{Z}_k$-teardrop Q can be expressed as the quotient of a smooth manifold by a compact Lie group.

Vector bundles over orbifolds can be defined locally as G-bundles.

One can take P to be the bundle of oriented orthonormal frames with respect to a metric, and then $P/SO(2) = Q$.

It is a conjecture of Adem and Ruan that all orbifolds can be expressed as a quotient of a manifold by a compact Lie group acting properly and almost freely.

This is known to be true for a large class of orbifolds.

Noneffective Orbifolds

An orbifold atlas for the orbifold Q consists of charts of the form $\{V, G, \pi\}$ where V is an open ball in $\mathbb{R}^n$ with (linear) G-action, and $\pi : V \to Q$ induces a homeomorphism of V/G onto its image in Q.

Orbifolds are **effective** if the group in each chart acts effectively on $\mathbb{R}^n$ and **noneffective** otherwise.

An (easy) example of a noneffective orbifold is a manifold equipped with the trivial action of a finite group G.

Noneffective Orbifolds

Noneffective orbifolds arise naturally as suborbifolds of effective orbifolds, as well as in other constructions for orbifolds.

Every *effective* orbifold does arise as the quotient of its orthonormal frame bundle (a smooth manifold) by $O(n)$.

It is not known whether every noneffective orbifold can be expressed as the quotient of a manifold by a compact Lie group.

A partial answer has been given by Henriques and Metzler (2003).

Orbifold Groupoids

Given an orbifold atlas, one defines a Lie groupoid $\mathcal{G}$ where

- The space of objects $G_0 = \coprod V$ is the disjoint union of the domains of the charts, and
- the space of arrows G_1 is given by the actions of the groups G on the charts as well as the fibered products $V_1 \times_Q V_2$ connecting points in distinct charts with the same image in Q.

The (underlying space) of Q is homeomorphic to the orbit space of the groupoid $\mathcal{G}$.

A Groupoid for the $\mathbb{Z}_k$-Teardrop

Orbifold Groupoids

The groupoid that arises from an orbifold atlas is a proper, étale Lie groupoid, and conversely the orbit space of every proper, étale Lie groupoid is an orbifold.

Definition

An **orbifold groupoid** is a proper, étale Lie groupoid.

Two orbifold groupoids $\mathcal{G}$ and $\mathcal{G}'$ present the same orbifold if and only if they are Morita equivalent.

If

$$\mathcal{G} \xrightarrow{\epsilon} \mathcal{H} \xleftarrow{\epsilon'} \mathcal{G}'$$

is a Morita equivalence of $\mathcal{G}$ and $\mathcal{G}'$ via the **orbifold** groupoid $\mathcal{H}$, then $\mathcal{H}$ is an atlas that refines those associated to $\mathcal{G}$ and $\mathcal{G}'$.

Orbifold Groupoids

Definition

An **orbifold** Q is the Morita equivalence class of an orbifold groupoid.

A groupoid in this equivalence class is a **presentation** of Q.

This definition clears up uncertainty about the appropriate definition of maps between orbifolds and the definition of a noneffective orbifold.

Note that a presentation $\mathcal{G}$ of Q may not be étale.

For instance, if Q is a quotient of a manifold M by a compact Lie group G, then $M \rtimes G$ is a presentation of Q.

The Inertia Orbifold

Let Q be an orbifold. The **inertia orbifold** $\tilde{Q}$ of Q is an orbifold associated to Q, consisting of pairs $(p, (g))$ with $p \in Q$, (g) the conjugacy class of $g \in G_p$.

If $Q = M/G$ is a global quotient, then $\tilde{Q}$ is given by

$$\coprod_{(g)} M^g / C_G(g),$$

(g) denotes the conjugacy class of g in G,

M^g the fixed-point set of g,

$C_G(g)$ the centralizer of g in G.

If Q is not a global quotient

A chart of the form $\{V, G, \pi\}$ for Q induces charts of the form $\{V^g, C_G(g), \pi_g\}$ for the inertia orbifold.

Example: If Q is the $\mathbb{Z}_k$-teardrop, then the inertia orbifold has k connected components.

In general, Q is a connected component of $\tilde{Q}$, the **nontwisted** sector, corresponding to the identity elements.

Other connected components are **twisted sectors**.

Example

In 2-dimensional orbifolds, singular points are called

cone points if they are fixed
by $\mathbb{Z}_k$ acting as rotations,

reflector lines if they are fixed
by $\mathbb{Z}_2$ acting as a reflection, and

corner reflectors if they are fixed
by a dihedral group D_{2n}.

Example

If Q is a 2-orbifold with reflector lines and corner reflectors, then the inertia orbifold consists of:

Q,

circles with trivial $\mathbb{Z}_2$-action,

quotients of such circles by $\mathbb{Z}_2$,

and points with trivial $\mathbb{Z}_n$- or $\mathbb{Z}_n \oplus \mathbb{Z}_2$-action.

The Inertia Orbifold

The inertia orbifold "plays the role of" the orbifold in many contexts.

(Kawasaki; 1981)

If Q is compact and effective, vector bundles over Q induce vector bundles over $\tilde{Q}$.

If D is an elliptic pseudodifferential operator on sections of bundles over Q, then D induces induces an operator $\tilde{D}$ on $\tilde{Q}$. Then

$$Ind_a(D) = Ind_{top}(D)$$

where the analytic index is defined in the usual way, and the topological index is defined as a cohomology class in $H^*(\tilde{Q})$.

The Inertia Orbifold

(Farsi; 1992)

The topological index, $Ind_{top}(D)$, of an elliptic pseudodifferential operator D on Q, defined on the inertia orbifold of Q, coincides with the analytic index of D as an element of $KK(C^*(Q), \mathbb{C})$.

The algebra $C^*(Q)$ is defined as $C(P) \rtimes G$ where P in the case that P is a smooth manifold, G a compact Lie group, and $Q = P/G$.

If Q is covered by a manifold M so that $Q = M/\Gamma$ with Γ discrete, then $C_0(M) \rtimes \Gamma$ is Morita equivalent to $C^*(Q)$.

The Inertia Orbifold

(Pflaum, Posthuma, Tang; 2008)

Given an orbifold cyclic Alexander-Spanier cochomology class $[f]$ of even order, the localized index $ind_{[f]}(D)$ of an elliptic operator D is equal to a topological index expressed in terms of the evaluation of characteristic classes on the inertia orbifold of the cotangent bundle $\widetilde{T^*Q}$.

This is a higher index theorem generalizing that of Connes-Moscovici to the orbifold setting.

The Inertia Orbifold

(Vafa, Witten, et. al.; 1985)

The role of the Euler characteristic for manifolds is replaced in models of string theory using orbifolds with the **stringy orbifold Euler characteristic**, the (topological) Euler characteristic of $\tilde{Q}$:

$$\begin{aligned}\chi_{orb}(M, G) &= \frac{1}{|G|} \sum_{gh=hg} \chi(M^{\langle g,h\rangle}) \\ &= \sum_{(g)} \chi(M^g / C_G(g)) \\ &= \chi_{top}(\tilde{Q}).\end{aligned}$$

The Inertia Orbifold

The rationalized orbifold K-theory of an effective orbifold Q is (additively) isomorphic to the singular cohomology of (the underlying topological space of) $\tilde{Q}$:

$$ch : K^*(Q) \otimes \mathbb{Q} \longrightarrow H^*(\tilde{Q}, \mathbb{Q}).$$

The K-theory of Q can be defined as the K-theory of $C^*(Q)$ for a description of Q as a quotient, or as $C^*(\mathcal{G})$ for an orbifold groupoid $\mathcal{G}$ presenting Q.

The Inertia Orbifold

(Lupercio, Uribe; 2002)

The loop space $\mathcal{L}Q$ of an orbifold Q presented by the orbifold groupoid $\mathcal{G}$ can be defined as the space of homomorphisms from S^1 (with a certain groupoid structure) to $\mathcal{G}$.

Then $\mathcal{L}Q$ is a topological groupoid that admits a natural S^1-action.

If Q is a manifold, then the space of fixed points under this S^1-action is homeomorphic to Q.

If Q is an orbifold, then the space of fixed points under this S^1-action is homeomorphic to $\tilde{Q}$.

Another example

Let $\mathbb{R}^4$ have standard basis $\{e_1, e_2, e_3, e_4\}$, and let the dihedral group $D_6 = \langle a, b \rangle$ act on $S^3 \subset \mathbb{R}^4$ as follows.

Let a denote the permutation $a = (123)$ on the basis elements.

Let

$$b = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix},$$

and let Q be presented by $S^3 \rtimes D_6$.

$\mathbb{Z}$-Sectors of $S^3 \rtimes D_6$

The inertia orbifold of $\tilde{Q}$ has three connected components:

$$\tilde{Q}_{(1)} = Q,$$

$$\tilde{Q}_{(a)} = \mathrm{Span}\{e_1 + e_2 + e_3, e_4\} \cap S^3 = S^1$$

(with trivial $\langle a \rangle = \mathbb{Z}_3$-action), and

$$\tilde{Q}_{(b)} = \mathrm{Span}\{e_1 + e_2, e_3\} \cap S^3 = S^1$$

(with trivial $\langle b \rangle = \mathbb{Z}_2$-action).

The two points fixed by all of D_6 do not appear.

In general the twisted sectors are finite singular covers of portions of the singular strata.

Only strata that are fixed by cyclic groups appear as separate strata.

Generalizations of the inertia orbifold have been introduced for global quotients by Bryan-Fulman and Tamanoi, and for general orbifolds by Chen-Ruan.

Γ-Sectors

Γ-Sectors

Let Γ be a (finitely generated, discrete) group, treated as a groupoid with one unit.

The space $\mathrm{HOM}(\Gamma, \mathcal{G})$ of groupoid homomorphisms inherits a manifold structure from $\mathcal{G}$ (with connected components of different dimensions).

The groupoid $\mathcal{G}$ acts on $\mathrm{HOM}(\Gamma, \mathcal{G})$ by conjugating homomorphisms pointwise.

The resulting translation groupoid $\mathrm{HOM}(\Gamma, \mathcal{G}) \rtimes \mathcal{G}$ is an orbifold groupoid.

Γ-Sectors

Definition (Γ-sectors)

The **orbifold of Γ-sectors of** Q is the orbifold $\tilde{Q}_\Gamma$ presented by

$$\mathcal{G}^\Gamma = \mathrm{HOM}(\Gamma, \mathcal{G}) \rtimes \mathcal{G}.$$

Given $\phi : \Gamma \to \mathcal{G}$ and a chart $\{V, G, \pi\}$ for Q, a neighborhood in $\mathrm{HOM}(\Gamma, \mathcal{G})$ is identified with $V^{\langle\phi\rangle}$ with $C_G(\phi)$-action.

The resulting orbifold is covered by charts of the form $\{V^{\langle\phi\rangle}, C_G(\phi), \pi_\phi\}$ where $C_G(\phi)$ is the centralizer of the image of ϕ in G.

Γ-Sectors

If Q is compact and Γ finitely generated, then the space of Γ-sectors is a disjoint union of compact orbifolds.

The connected component consisting of trivial homomorphisms is diffeomorphic to Q.

Lemma

A Morita equivalence between orbifold groupoids $\mathcal{G}$ and $\mathcal{H}$ induces a Morita equivalence between $\mathcal{G}^\Gamma$ and $\mathcal{H}^\Gamma$.

Example: $\mathbb{F}_2$-Sectors of $S^3 \rtimes D_6$

The space of $\mathbb{F}_2$-sectors of Q has fourteen connected components.

One is diffeomorphic to $\tilde{Q}_{(1)} = S^3 \rtimes D_6$
(corresponding to homomorphisms with trivial image),

Four are diffeomorphic to $\tilde{Q}_{(a)} = S^1 \rtimes \mathbb{Z}_3^{triv}$
(corresponding to homomorphisms with image $\langle a \rangle$),

Three are diffeomorphic to $\tilde{Q}_{(b)} = S^1 \rtimes \mathbb{Z}_2^{triv}$
(corresponding to homomorphisms with image $\langle b \rangle$),

Six are diffeomorphic to $\{pt\}$ (with no group action)
(corresponding to homomorphisms with image D_6).

Special cases of the Γ-sectors

The inertia orbifold corresponds to the case $\Gamma = \mathbb{Z}$.

The k-multi-sectors of Chen-Ruan (2001) correspond to the case $\Gamma = \mathbb{F}_k$.

For global quotients, the generalized orbifold Euler characteristics of Bryan-Fulman (1997) correspond to the (topological) Euler characteristics of the $\mathbb{Z}^m$-sectors.

For global quotients, the generalized orbifold Euler characteristics of Tamanoi (2001) correspond to the Euler-Satake characteristics of the Γ-sectors.

Other Properties of the Γ-Sectors

Iterating the construction corresponds to using a product group Γ.

That is,

$$\left(\mathcal{G}^{\Gamma_1}\right)^{\Gamma_2} \overset{iso}{\cong} \mathcal{G}^{\Gamma_1 \times \Gamma_2}$$

as groupoids.

Hence,

$$\widetilde{(\tilde{Q}_{\Gamma_1})}_{\Gamma_2} \overset{diffeo}{\cong} \tilde{Q}_{\Gamma_1 \times \Gamma_2}$$

as orbifolds.

Other Properties of the Γ-Sectors

It follows that $\tilde{Q}_{\mathbb{Z}^2} = \widetilde{(\tilde{Q})}$, $\tilde{Q}_{\mathbb{Z}^3} = \widetilde{\widetilde{(\tilde{Q})}}$, etc.

The *m*th orbifold Euler characteristic of Bryan and Fulman (corresponding to the $\mathbb{Z}^m$-sectors) is the Euler characteristic of the orbifold formed by applying the inertia construction m times to Q.

Theorem (S., 2004)

If Q is a closed orbifold, then the topological Euler characteristic of the underlying space of Q coincides with the Euler-Satake characteristic of the inertia orbifold of Q:

$$\chi_{top}(Q) = \chi_{ES}(\tilde{Q}).$$

As a result, the Euler characteristic of the underlying topological space and the Euler-Satake characteristic are connected by:

$$\chi_{top}\left(\tilde{Q}_\Gamma\right) = \chi_{ES}\left(\tilde{Q}_{\Gamma\times\mathbb{Z}}\right).$$

Applications of the Γ-Sectors

Applications of the Γ-Sectors

A group Γ **covers the local groups of** Q if there is a surjective homomorphism from Γ onto every isotropy group of Q.

Theorem (Farsi, S.; 2008)

Let Q be a closed (codimension-2) orbifold and Γ a group that covers the local groups of Q. Then Q admits a smooth, nonvanishing vector field if and only if $\chi_{ES}\left(\tilde{Q}_{(\phi)}\right) = 0$ for each Γ-sector $\tilde{Q}_{(\phi)}$.

In the case that Q is oriented, this is equivalent to the Euler class of $\tilde{Q}_\Gamma$ in $H^*(\tilde{Q}_\Gamma)$ vanishing.

For every compact orbifold, there is a finite group that covers the local groups.

New Orbifold Invariants

Vector bundles, metrics, differential operators, etc. over Q induce the same over $\tilde{Q}_\Gamma$.

Since the sectors are generally nonreduced orbifolds, these bundles may be so-called *bad orbifold vector bundles*; i.e. bundles where group elements that act trivially on the base space do not act trivially on the fibers.

As an example, if M is a manifold with $\mathbb{Z}_k$ acting trivially, then the quotient of the $\mathbb{Z}_k$-bundle given by $M \times \mathbb{R}^2$ with $\mathbb{Z}_k$ acting as rotations on $\mathbb{R}^2$, is a bad orbifold vector bundle.

The usual description of characteristic classes in de Rham cohomology fails in this case, as the bundles do no admit nonzero sections.

New Orbifold Invariants

Theorem (S., 2007)

For every bad orbifold vector bundle $\rho : E \to Q$, there is an orbifold R of which Q is a suborbifold and a good orbifold vector bundle $\hat{\rho} : \hat{E} \to R$ such that E is isomorphic to the restriction $\hat{E}|_Q$ of $\hat{E}$ to Q.

By lifting to $\hat{E}$, this construction can be used to extend the definitions of characteristic classes to arbitrary orbifold vector bundles.

Orbifold invariants defined on Q can be applied to the Γ-sectors to yield new orbifold invariants for each Γ.

Do these new invariants carry additional information?

Basic Example

(Satake, 1957) The Euler-Satake characteristic is an invariant associated to the Euler operator (the "top term" of the topological index).

Definition

For each finitely generated group Γ, the Γ-**Euler-Satake characteristic** is

$$\chi_\Gamma^{ES}(Q) = \chi_{ES}\left(\tilde{Q}_\Gamma\right),$$

the Euler-Satake characteristic of the Γ-sectors of Q.

If Q is a manifold, then $\chi_\Gamma^{ES} = \chi(Q)$ for each Γ.

Classifying Closed, Orientable 2-orbifolds

Theorem (DuVal, Schulte, S., Taylor; 2009)

Let Q and Q' be closed, effective, orientable 2-orbifolds such that $\chi_{\mathbb{Z}^l}^{ES}(Q) = \chi_{\mathbb{Z}^l}^{ES}(Q')$ for an unbounded set of nonnegative integers l. Then Q and Q' are diffeomorphic.

Theorem (DuVal, Schulte, S., Taylor; 2009)

Let $N \geq 2$ be an integer and let $\mathfrak{G}$ be any finite collection of finitely generated discrete groups. Then there are distinct closed, connected, effective, orientable 2-orbifolds $Q_1, Q_2, \ldots, Q_N$ such that for each $\Gamma \in \mathfrak{G}$,

$$\chi_\Gamma^{ES}(Q_1) = \chi_\Gamma^{ES}(Q_2) = \cdots = \chi_\Gamma^{ES}(Q_N).$$

Nonorientable 2-orbifolds

Theorem (Schulte, S., Taylor; 2009)

Let Q and Q' be closed, effective 2-orbifolds such that $\chi_{\mathbb{Z}^l}^{ES}(Q) = \chi_{\mathbb{Z}^l}^{ES}(Q')$ for an unbounded set of nonnegative integers l and $\chi_{\mathbb{F}_k}^{ES}(Q) = \chi_{\mathbb{F}_k}^{ES}(Q')$ for an unbounded set of nonnegative integers k. Then Q and Q' have underlying spaces with the same Euler characteristic, the same number of cone points and corner reflectors with the same orders.

Any collection of free and free abelian Γ that determines this information contains free groups of arbitrarily large rank and free abelian groups of arbitrarily large rank.

Computations for Wreath Symmetric Products

Wreath Symmetric Products

Let G be a group and $\mathcal{S}_n$ denote the symmetric group on n elements. Then $\mathcal{S}_n$ acts on G^n by setting

$$\sigma(g_1, g_2, \ldots, g_n) = (g_{\sigma^{-1}(1)}, g_{\sigma^{-1}(2)}, \cdots, g_{\sigma^{-1}(n)})$$

for $\sigma \in \mathcal{S}_n$ and $(g_1, \ldots, g_n) \in G^n$.

The wreath product $G(\mathcal{S}_n)$ of G by $\mathcal{S}_n$ is the semidirect product of G^n by this action.

Wreath Symmetric Products

Definition

Let G be a compact Lie group acting locally freely on a smooth manifold M so that $M \rtimes G$ is Morita equivalent to an orbifold groupoid, and hence presents an orbifold Q.

Let $G(\mathcal{S}^n)$ act on M^n via

$$((g_1, \ldots, g_n), \sigma)(x_1, \ldots, x_n) = (g_1 x_{\sigma^{-1}(1)}, \ldots, g_n x_{\sigma^{-1}(n)}).$$

Let

$$MG(\mathcal{S}^n) = M^n \rtimes G(\mathcal{S}^n).$$

Then $MG(\mathcal{S}^n)$ is Morita equivalent to an orbifold groupoid and presents the **nth wreath symmetric product of Q.**

Examples

If $M = S^1$, $n = 2$, and $G = \mathbb{Z}_2$ acting as a reflection, then $MG(\mathcal{S}_2)$ is a disk with reflector lines as the boundary and three corner reflectors with isotropy D_8.

Orbifold Wreath Product Ring

Let

$$\mathcal{F}_G^q(M) = \bigoplus_{n\geq 0} q^n \underline{K}_{G(\mathcal{S}_n)}(M^n).$$

Define a product on $\mathcal{F}_G(M)$ by the composition of the Induction map and the Künneth isomorphism:

$$\underline{K}_{G(\mathcal{S}_n)}(M^n)\otimes\underline{K}_{G(\mathcal{S}_m)}(M^m) \stackrel{\text{Kün}}{\to} \underline{K}_{G(\mathcal{S}_n)\times G(\mathcal{S}_m)}(M^{n+m}) \stackrel{Ind}{\to} \underline{K}_{G(\mathcal{S}_{n+m})}(M^{n+m}).$$

Define a coproduct by

$$\underline{K}_{G(\mathcal{S}_n)}(M^n) \stackrel{Res}{\to} \bigoplus_{m=0}^{n} \underline{K}_{G(\mathcal{S}_m)\times G(\mathcal{S}_{n-m})}(M^n) \stackrel{\text{Kün}^{-1}}{\to} \underline{K}_{G(\mathcal{S}_n)}(M^n)\otimes\underline{K}_{G(\mathcal{S}_m)}(M^m).$$

Orbifold Wreath Product Ring

Theorem (Farsi, S., 2009)

The $(\mathbb{Z}_+, \mathbb{Z}_2)$-graded algebra $\mathcal{F}^q_G(M)$ is isomorphic to the supersymmetric algebra

$$\mathcal{SS}\left[\bigoplus_{n\geq 1} q^n \underline{K}_G(M)\right] = \mathcal{SP}\left[\bigoplus_{n\geq 1} q^n \underline{K}^0_G(M)\right] \otimes \Lambda \left[\bigoplus_{n\geq 1} q^n \underline{K}^1_G(M)\right]$$

and hence inherits the structure of a Hopf algebra.

This generalizes results of Segal (1996) for the case of symmetric products (i.e. G trivial) and Wang (2001) for the case of G finite to arbitrary effective orbifolds.

Generating Functions for Wreath Symmetric Products

Theorem (Farsi, S.; 2009)

If G is a compact Lie group acting locally freely on the compact manifold M, then

$$\sum_{n \geq 0} q^n \chi_{ES}^{\Gamma}(M^n \rtimes G(\mathcal{S}_n)) = \exp\left\{ \sum_{r \geq 1} \frac{q^r}{r} \left[\sum_{H : |\Gamma/H| = r} \chi_H^{ES}(M \rtimes G) \right] \right\},$$

where H ranges over subgroups of Γ with finite index r.

This generalizes results of Tamanoi for global quotient orbifolds (i.e. G finite).

Thank you!