Some remarks on Kadison-Singer Problem

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Let $\mathcal{H}$ be a separable Hilbert space, $B(\mathcal{H})$ the W^*-algebra of all bounded linear operators on $\mathcal{H}$.
(ξ_n), a fixed orthonormal basis of $\mathcal{H}$.
$p_n \xi_n = \xi_n$ $(n = 1, 2, \cdots)$: p_n, a family of mutually orthogonal one-dimensional projections.
$\mathcal{C}$; an atomic maximal commutative W^*-subalgebra generated by $\{p_n\}$
P: a projection of $B(\mathcal{H})$ onto $\mathcal{C}$ given by
$P(a) = \sum_{n=1}^{\infty} (a\xi_n, \xi_n) p_n \quad (a \in B(\mathcal{H}))$.

$\mathcal{C} = C(\beta N)$, where βN is the Stone-Čech compactification of N. For $t \in \beta N$, put $P(a)(t) = \varphi_t(a)$; then by Anderson's theorem, φ_t is a pure state on $B(\mathcal{H})$. If $t \in N$, then φ_t is a unique pure state extension of $\varphi_t | \mathcal{C}$ to $B(\mathcal{H})$.

> Kadison-Singer Problem. For $t \in \beta N \setminus N$, can we conclude that $\varphi_t | \mathcal{C}$ has a unique pure state extension φ_t to $B(\mathcal{H})$?

<u>Theorem 1.</u> The following properties are equivalent

(1) Kadison-Singer problem is positive for all $t \in \beta N \setminus N$

(2) for any commutative AW^*-subalgebra D of $B(H)^{**}$ with $C \subset D \subset B(H)^{**}$ and let Q be a norm-one projection of $B(H)^{**}$ onto D (always exists); then $Q(a) = P(a)$ for $a \in B(H)$.

(3) $V(a) \cap C = \{P(a)\} \quad (a \in B(H))$, where $V(a)$ is the norm-closed convex subset of $B(H)$ generated by $\{uau^* \mid u \in C^u\}$ (C^u is the set of all unitary elements of C.

(4) $V(a)^{oo} \cap C' = \{P(a)\}$, where $V(a)^{oo}$ is the bipolar of $V(a)$ in $B(H)^{**}$ and C' is the commutant of C in $B(H)^{**}$.

Proof. <u>(1) $\Rightarrow$ (2)</u> Since the spectrum space of D is Stonean, there is a norm-one projection of $B(H)^{**}$ onto D. Let $D = C(K)$ and suppose $P(a) \neq Q(a)$ for some $a \in B(H)$; then $\exists\, t_0 \in K$: $P(a)(t_0) \neq Q(a)(t_0)$. On the other hand, $P(c)(t_0) = c(t_0) = Q(c)(t_0)$ for $c \in C$; hence by (1), $P(a)(t_0) = Q(a)(t_0)$, a contradiction.

<u>(2) $\Rightarrow$ (4).</u> By Markov-Kakutani fixed

point theorem, for $b \in B(H)^{\Delta}$, $V(b)^{oo} \cap C' \neq \phi$. Take $b_0 \in V(b)^{oo} \cap C'$; then $b_0 \in (B(H)^{**})^{\Delta}$.

Let D be the commutative W* subalgebra of $B(H)^{**}$ generated by b_0 and C. Since $\| v b v^* + d \| = \| (v(b+d) v^* \| = \| b + d \|$ for $d \in D$ and $v \in C^u$. Hence $\| b_0 + d \| \leqq \| b + d \|$. Define $T(\lambda b + d)$

$= \lambda b_0 + d$ ($\lambda \in \mathbb{C}$, $d \in D$); then by the extension property of D, T can be extended to a norm-one projection $\tilde{T}$ of $B(H)^{**}$ onto D. By (2), $\tilde{T}(b) = b_0 = P(b)$; hence $V(b)^{oo} \cap C' = \{P(b)\}$.

For $a \in B(H)$, let $a = a_1 + i a_2$ ($a_1, a_2 \in B(H)^{\Delta}$); $a_0 \in V(a)^{oo} \cap C'$ implies $\frac{a_0 + a_0^*}{2} \in V(a_1)^{oo} \cap C$ and $\frac{i a_0 - i a_0^*}{2} \in V(a_2)^{oo} \cap C$. Hence $a_0 = P(a)$.

<u>(4) ⇒ (3)</u> Since $P(a) \in V(a)^{oo}$ ($a \in B(H)$), there is a directed set of elements $\{x_\alpha\}$ in $V(a)$ such that $\sigma(B(H), B(H)^*)$-$\lim_\alpha x_\alpha = P(a)$. Therefore by the convexity of $V(a)$, there is a sequence $\{y_n\}$ in $V(a)$ such that $\{y_n\}$ converges to $P(a)$ in norm; hence $P(a) = V(a) \cap C$.

<u>(3) ⇒ (1)</u> Suppose φ is a state on $B(H)$ such that $\varphi(c) = c(t)$ for $c \in C$ ([redacted] $|\varphi(a(c - c(t)1))| \leqq \varphi(a a^*)^{1/2} \varphi((c - c(t)1)^*(c - c(t)))^{1/2}$ $= 0$; hence $\varphi(ac) = \varphi(ca) = c(t)\varphi(a)$

Hence $\varphi(u a u^*) = \varphi(a)$ ($a \in B(H)$, $u \in C^u$) and so $\varphi(a) = P(a)$

Remark. Property (4) in Theorem 1 is somewhat surprising to me and the author inclines to the view that the problem has a negative solution as the proposers thought in 1959 or at least we might need one more axiom to get a positive solution.

One can easily show that $V(a)^{oo} \cap C^{oo} = \{p(a)\}$ if $V(a)^{oo} \cap C^{oo} \neq \emptyset$. In fact for $y \in V(a)^{oo} \cap C^{oo}$, there is a directe set of $\{\sum \lambda_{j\alpha} u_{j\alpha}\, a\, u_{j\alpha}^* \mid \lambda_{j,\alpha} \geq 0,\ \sum_{j=1}^{n_\alpha} \lambda_{j,\alpha} = 1,\ u_{j,\alpha} \in C^u\}$ such that $\sigma(B(\mathcal{H})^{**}, B(\mathcal{H})^*)$-$\lim_\alpha \sum \lambda_{j,\alpha} u_{j\alpha}\, a\, u_{j\alpha}^* = y$; hence $p^{**}(\sum \lambda_{j\alpha} u_{j\alpha}\, a\, u_{j\alpha}^*) = p^{**}(a) \to p^{**}(y)$ and so $p^{**}(a) = p(a) = p^{**}(y)$.

Since $y \in C^{oo}$, $p^{**}(y) = y$ and so $y = p(a)$.

Therefore If $C^{oo} = C'$ in $B(\mathcal{H})^{**}$ (namely C^{oo} is a masa in $B(\mathcal{H})^{**}$), then $V^{oo} \cap C^{oo} = \{p(a)\}$ and so $V \cap C = \{p(a)\}$. Hence we can conclude that K-S is true for all $t \in \beta N \setminus N$.

However the recent result of Akemann and Weaver implies that under the assumption of the continuum hypothesis, $C^{co} \subsetneqq C'$ in $B(H)^{**}$. This fact will be explained later. I do not know whether one can prove $C^{oo} \subsetneqq C'$ in $B(H)^{**}$ within ZFC.

Therefore the following problem would be interesting.

Problem 1. Can we conclude that $C^{oo} \subsetneqq C'$ in $B(H)^{**}$ within ZFC ?

If $C^{oo} = C'$ in $B(H)^{**}$ is consistent to ZFC, then the statement of "K-S is true for all $t \in \beta N$" is consistent to ZFC.

Properties of P

P is a norm-one projection of $B(\mathcal{H})$ onto C satisfying the following properties

(i) P is $\sigma(B(\mathcal{H}), T(\mathcal{H}))$-continuous

(ii) $P(a^*a) \geq 0$ and $P(a^*a) = 0$ if and only if $a = 0$ $(a \in B(\mathcal{H}))$

(iii) $P(1) = 1$

(iv) $P(ac) = P(ca) = cP(a)$ $(c \in C,\ a \in B(\mathcal{H}))$

(v) $W(a) \cap C = \{P(a)\}$, where $W(a)$ is the $\sigma(B(\mathcal{H}), T(\mathcal{H}))$-closed convex subset of $B(\mathcal{H})$ generated by $\{uau^* \mid a \in B(\mathcal{H}),\ u \in C^u\}$

Proof. (i) $P(a) = \sum_{n=1}^{\infty} (a\xi_n, \xi_n)\, p_n$ with $f = \sum \lambda_n p_n$

suppose $a_\alpha \to 0$ $(\sigma(B(\mathcal{H}), T(\mathcal{H})))$, $\|a_\alpha\| \leq 1$ $\Rightarrow$ For $f \in T(\mathcal{H})$,

$|\tau(P(a_\alpha)f)| = |\sum_{n=1}^{\infty} \lambda_n (a_\alpha \xi_n, \xi_n)| < |\sum_{n=1}^{m} \lambda_n (a_\alpha \xi_n, \xi_n)| + \varepsilon$

; hence $\overline{\lim_\alpha}\, |\tau(P(a_\alpha)f)| \leq \varepsilon$ and so

$\tau(P(a_\alpha)f) \to 0$, where $f = \sum \lambda_n p_n$ and $\sum |\lambda_n| < +\infty$

(ii) $P(a^*a) = \sum \|a\xi_n\|^2 p_n = 0 \Rightarrow a\xi_n = 0$ $(n = 1, 2, \cdots)$

; hence $a = 0$.

(v) $W(a) \cap C' \neq \phi$ by Markov-Kakutani fixed point theorem. $C = C'$ in $B(\mathcal{H})$; hence $W(a) \cap C \neq \phi$

For $a_0 \in W(a) \cap C$, $\exists \{\sum \lambda_{j,\alpha} u_{j,\alpha} a u_{j,\alpha}^*\} \subset W(a)$

$\sum \lambda_{j,\alpha} u_{j,\alpha} a u_{j,\alpha}^* \to a_0$ $(\sigma(B(\mathcal{H}), T(\mathcal{H})))$

Hence $P(\sum \lambda_j u_j a u_j^*) = P(a) \to P(a_0)$; hence $P(a) = P(a_0)$. Since $a_0 \in C$, $P(a_0) = a_0$; hence $P(a) = a_0$ $\perp$

Let P^{**} be the second dual of P; then P^{**} is a norm-one $\sigma(B(\mathcal{H})^{**}, B(\mathcal{H})^*)$-continuous projection of $B(\mathcal{H})^{**}$ onto $C^{\circ\circ}$ satisfying the following properties.

(i) $P^{**}(a) = P(a) \quad (a \in B(\mathcal{H}))$

(ii) $P^{**}(x^*x) \geqq 0 \quad (x \in B(\mathcal{H})^{**})$

(iii) $P^{**}(cx) = P^{**}(xc) = cP^{**}(x) \quad (c \in C,\ x \in B(\mathcal{H})^{**})$.

Now assume that K-S problem is true for all $t \in \beta N \setminus N$. Then we have the following Theorem.

Theorem 2. $P^{**}(d_1 x d_2) = P^{**}(d_1) P^{**}(x) P^{**}(d_2)$ for $d_1, d_2 \in \mathcal{D}$ and $x \in B(\mathcal{H})^{**}$, where $\mathcal{D} = C' \cap B(\mathcal{H})^{**}$.

To prove we shall provide some lemmas.

Lemma 1. $P^{**}(u' a u'^*) = P^{**}(a) = P(a)$ $(a \in B(\mathcal{H}),\ u' \in \mathcal{D}^u)$.

Proof. For $u \in C^u$, $uu'au'^*u^* = u'(uau^*)u'^*$; hence $V(u'au'^*) = u'V(a)u'^*$, where $V(a)$ is the norm-closed convex subset of $B(\mathcal{H})$ generated by $\{uau^* \mid u \in C^u\}$ for $a \in B(\mathcal{H})$ and $V(u'au'^*)$ is the norm closed convex subset of $B(\mathcal{H})^{**}$ generated by $\{uu'au'^*u^* \mid u \in C^u\}$ for $u'au'^*$.

Therefore $V(u'au'^*) \cap C^{oo} = u'V(a)u'^* \cap C^{oo}$

$= u'(V(a) \cap C^{oo})u'^* = u'p(a)u'^* = p(a)$ and so $p^{**}(u'xu'^*) = p^{**}(x)$ for $x \in B(\mathcal{H})^{**}$ $\perp$

Lemma 2. $p^{**}(dx) = p^{**}(xd)$ for $x \in B(\mathcal{H})^{**}$ and $d \in \mathcal{D}$.

Proof $p^{**}(u'x) = p^{**}(u'^*u'xu') = p^{**}(xu')$

$(u' \in \mathcal{D}^u$, $x \in B(\mathcal{H})^{**})$; hence $p^{**}(dx) = p^{**}(xd)$ $\perp$

Lemma 3. $p^{**}(d_1xd_2) = p^{**}(d_1)p^{**}(x)p^{**}(d_2)$

$(d_1, d_2 \in \mathcal{D}$ and $x \in B(\mathcal{H})^{**})$.

Proof. For $h\ (>0) \in \mathcal{D}$, $p^{**}(hx) = p^{**}(h^{1/2}xh^{1/2})$; hence $p^{**}(hx^*x) \geqq 0$.

Now $p^{**}(hc) = p^{**}(h)c$ for $c \in C^{oo}$ and $p^{**}(ha) \geqq 0$ for $a\ (\geqq 0) \in B(\mathcal{H})$.

Let $C^{oo} = C(\Omega)$, where Ω is a compact Hausdorff space and $C(\Omega)$ is the C^*-algebra of all continuous functions on Ω. $p^{**}(hc)(\Delta) = p^{**}(h)(\Delta)\, c(\Delta)$ for $\Delta \in \Omega$ and $p^{**}(ha)(\Delta) = p^{**}(h^{1/2} a h^{1/2})(\Delta) \geq 0$ for $a (\geq 0) \in B(\mathcal{H})$. Suppose $p^{**}(h)(\Delta) > 0$; then

$$\frac{p^{**}(hc)(\Delta)}{p^{**}(h)(\Delta)} = \frac{p^{**}(h)(\Delta)\, c(\Delta)}{p^{**}(h)(\Delta)} = c(\Delta) \quad \text{and}$$

$$\frac{p^{**}(ha)(\Delta)}{p^{**}(h)(\Delta)} \quad \text{is a state on } B(\mathcal{H}).$$

Since $c \longrightarrow c(\Delta)$ $(c \in C)$ is a character of C, there is a point t in βN such that $c(\Delta) = c(t)$ for $c \in C$ and so the unicity of extension implies $\dfrac{p^{**}(ha)(\Delta)}{p^{**}(h)(\Delta)} = p(a)(t)$ $(a \in B(\mathcal{H}))$.

Since $p(a) \in C$, $p^{**}(ha)(\Delta) = p^{**}(h)(\Delta)\, p(a)(\Delta)$.

If $p^{**}(h)(\Delta) = 0$, then $|p^{**}(ha)(\Delta)| \leq p^{**}(h)(\Delta)^{1/2}\cdot$ $p^{**}(a^* h a)^{1/2} = 0$; hence $p^{**}(ha)(\Delta) =$ $p^{**}(h)(\Delta)\, p(a)(\Delta)$ for all $\Delta \in \Omega$.

Hence $p^{**}(ha)(\Delta) = p^{**}(h)(\Delta)\, p(a)(\Delta)$ and so $p^{**}(da) = p^{**}(d)\, p(a)$ $(d \in \mathcal{D},\ a \in B(\mathcal{H}))$

By the σ-weak continuity of p^{**}, we have

$p^{**}(dx) = p^{**}(d)\, p^{**}(x)$ for $x \in B(\mathcal{H})^{**}$ and $d \in \mathcal{D}$. Hence $p^{**}(d_1 x d_2) = p^{**}(d_1)\, p^{**}(x d_2)$ $= p^{**}(d_1)\,(p^{**}(d_2^* x^*))^* = p^{**}(d_1)\, p^{**}(x)\, p^{**}(d_2)$ $\perp$

By Theorem 2, the restriction of P^{**} to $\mathcal{D}$ is a σ-weakly continuous $*$-homomorphism of $\mathcal{D}$ onto C^{00}. Hence there exists a central projection q of $\mathcal{D}$ such that $\ker(P^{**}|\mathcal{D}) = \mathcal{D}q$ and so $P^{**}(q)=0$. Put $p = 1-q$; then

$$\mathcal{D} = C^{00} + \mathcal{D}q = C^{00}p \oplus \mathcal{D}q \quad \text{and} \quad \mathcal{D}p = C^{00}p.$$

Moreover $P^{**}|C^{00}p$ is a $*$-isomorphism of $C^{00}p$ onto C^{00}, and $P^{**}(p)=1$.

Hence we have

Theorem 3. There exist central projections p, q of $\mathcal{D} = C' \cap B(\mathcal{H})^{**}$ such that $\mathcal{D}q$ is a $\sigma(B(\mathcal{H})^{**}, B(\mathcal{H})^{*})$-closed ideal of $\mathcal{D}$ which is the kernel of $P^{**}|\mathcal{D}$ and $P^{**}(q)=0$.

Moreover

$$\mathcal{D} = C^{00} + \mathcal{D}q = C^{00}p + \mathcal{D}q \quad \text{and} \quad \mathcal{D}p = C^{00}p$$

and $P^{**}|C^{00}p$ is a $*$-isomorphism of $C^{00}p$ onto C^{00}, and $p = 1-q$.

Theorem 4. Let $\varphi \in \mathfrak{S}(C)$ and put $\tilde{\varphi}(a) = \varphi(P(a))$ $(a \in B(\mathcal{H}))$. Let $s(\varphi)$ (resp $s(\tilde{\varphi})$) be the support of φ in C^{00} (resp. $\tilde{\varphi}$ in $B(\mathcal{H})^{**}$); then $s(\tilde{\varphi}) = s(\varphi)p$.

Proof. Since $\tilde{\varphi}(uau^*) = \tilde{\varphi}(a)$ $(u \in C^*, a \in B(H))$, $\Delta(\tilde{\varphi}) \in D$. On the other hand $\Delta(\varphi) = 1 - \sup_{\substack{e \in C_p^{oo} \\ \varphi(e)=0}} e$, where C_p^{oo} is the set of all projections in C^{oo}: $\hat{\varphi}(\Delta(\varphi)p) = \varphi(p^{**}(\Delta(\varphi)p))$

$= \varphi(\Delta(\varphi)) = 1$; hence $\Delta(\tilde{\varphi}) \leq \Delta(\varphi)p$.

For $r \in D^p$ with $r < \Delta(\varphi)p$, where D^p is the set of all projections of D, we have $r = rp = p^{**}(r)p <$ $p^{**}(\Delta(\varphi))p$. Since $C^{oo}p$ is $*$-isomorphic to C^{oo} under p^{**}, $p^{**}(r) < p^{**}(\Delta(\varphi)) = \Delta(\varphi)$; hence $\varphi(p^{**}(r)) = \tilde{\varphi}(r) < 1$ ┘

Theorem 4. For $t \in \beta N$, let $\varphi_t(c) = c(t)$ $(c \in C)$ and $\tilde{\varphi}_t(a) = \varphi_t(p(a))$ $(a \in B(H))$; then $\Delta(\tilde{\varphi}_t) = \Delta(\varphi_t)$ and so $\Delta(\varphi_t) < p$

Proof. Since φ_t has a unique state extension to $B(H)$, $\{c \in C \mid \varphi_t(c^*c) = 0\} = \mathcal{L}_t$ is a maximal ideal of C and the norm closure of $B(H)\mathcal{L}_t$ is a maximal left ideal of $B(H)$; hence $B(H)^{**}(1-\Delta(\varphi_t))$ is a maximal σ-closed left ideal of $B(H)^{**}$.

Therefore $\Delta(\varphi_t) = \Delta(\tilde{\varphi}_t)$ is a one-dimensional projection in $B(H)^{**}$; hence $\Delta(\tilde{\varphi}_t) = \Delta(\varphi_t) < p$ and so $\Delta(\tilde{\varphi}_t) \in C^{oo}$ ┘

Let u be a unitary element in $B(\mathcal{H})$ such that $uCu^* = C$. Define $\alpha_u(c) = ucu^*$ $(c \in C)$; then α_u is a $*$-automorphism of C. Conversely if α is a $*$-automorphism of C, $\alpha(p_n) = p_{\pi(n)}$; then a permutation π is defined on N. Define $u\xi_n = \xi_{\pi(n)}$; then u is a unitary element of $B(\mathcal{H})$ and $up_nu^* = p_{\pi(n)}$ $(n \in N)$. Therefore all $*$-automorphisms of C are defined by unitary elements of $B(\mathcal{H})$ such that $uCu^* = C$.

Moreover there exists a one-to-one correspondence between the automorphism group of C and the homeomorphism group of βN.

Theorem 5. For $u \in B(\mathcal{H})^u$ with $uCu^* = C$, let $\alpha_u(a) = uau^*$ $(a \in B(\mathcal{H}))$; then $\alpha_u P = P\alpha_u$ on $B(\mathcal{H})$.

Proof. $\alpha_u P(a) = \alpha_u(\sum (a\xi_n, \xi_n)p_n) = \sum (a\xi_n, \xi_n) u p_n u^*$
$= \sum (a\xi_n, \xi_n) p_{\pi(n)}$ and

$$P\alpha_u(a) = \sum (uau^*\xi_n, \xi_n) p_n = \sum (a\xi_{\pi^{-1}(n)}, \xi_{\pi^{-1}(n)}) p_n$$

$$= \sum (a\xi_n, \xi_n) p_{\pi(n)} \quad \lrcorner$$

Corollary 1. For $u \in B(\mathcal{H})^u$ with $uCu^* = C$, $\alpha_u P^{**} = P^{**}\alpha_u$ on $B(\mathcal{H})^{**}$.

Corollary 2. For $u \in B(\mathcal{H})^u$ with $uCu^* = C$, $upu^* = p$ and $uqu^* = q$, where $\mathcal{D} = C''p \oplus \mathcal{D}q$.

Proof. Since $uCu^* = C$, $u\mathcal{D}u^* = \mathcal{D}$

$\alpha_u P^{**}(p) = uu^* = 1 = P^{**}\alpha_u(p) = P^{**}(upu^*)$; hence $upu^*p = P^{**}(upu^*)p = p$. Therefore $upu^* \geq p$

Since $uCu^* = C$ implies $u^*Cu = C$, $upu^* \geqq p$; hence $u^*pu = p$. $\lrcorner$

Let G be the group of all homeomorphisms on βN and for $t \in \beta N$, let O_t be the orbit $Gt \subset \beta N$.

Let z_t be the central envelop of $\Delta(\varphi_t)$ in $B(H)^{**}$. Then $\Delta(\varphi_s) \leqq z_t$ for all $s \in O_t$, because $u\Delta(\varphi_t)u^* = \Delta(\varphi_s)$ for some $u \in B(H)^u$ with $uCu^* = C$ and so $u\Delta(\varphi_t)u^* \leqq uz_tu^* = z_t$.

Since $upu^* = p$, $\Delta(\varphi_s) \leqq p$ for all $s \in O_t$. Therefore $\sum_{s \in O_t} \Delta(\varphi_s) \leqq z_t p = p^{**}(z_t)p$ and

$\sum_{s \in O_t} \Delta(\varphi_s) \in C^{**}$.

Theorem 6. $\boxed{\text{Card}(O_t) \leqq 2^{\aleph_0}}$

Proof. Since $\{\Delta(\varphi_s) \mid s \in O_t\}$ is a family of mutually orthogonal one-dimensional projections in $B(H)^{**}z_t$, $\text{Card}(O_t) \leqq \dim(z_t)$.
On the other hand, $B(H)^{**}z_t$ is $*$-isomorphic to an irreducible representation $\{\pi_{\tilde\varphi_t}, H_{\tilde\varphi_t}\}$ of $B(H)$ constructed via $\tilde\varphi_t$. Since $\dim(H_{\tilde\varphi_t}) \leqq \text{Card}(B(H))$, $\dim(H_{\tilde\varphi_t}) \leqq 2^{\aleph_0}$ $\lrcorner$

Theorem 7. $C^{oo}p$ is a masa in $pB(\mathcal{H})^{**}p$ and $pP^{**}|pB(\mathcal{H})^{**}p$ is a σ-continuous norm-one projection of $pB(\mathcal{H})^{**}p$ onto $C^{oo}p$

Proof. $p \in \mathcal{D}$ and so $(p\mathcal{D}p)' \cap pB(\mathcal{H})^{**}p =$ $(C^{oo}p)' \cap pB(\mathcal{H})^{**}p = \mathcal{D} \cap pB(\mathcal{H})^{**}p = p\mathcal{D}p = C^{oo}p$. Therefore $C^{oo}p$ is a masa in $pB(\mathcal{H})^{**}p$. Moreover $p \cdot P^{**}(pB(\mathcal{H})^{**}p) = pC^{oo} = C^{oo}p$; hence $p \cdot P^{**}|pB(\mathcal{H})^{**}p$ is a σ-continuous norm one projection of $pB(\mathcal{H})^{**}p$ onto $C^{oo}p$ $\lrcorner$

Theorem 8. For $t \in \beta N$, $B(\mathcal{H})^{**}z_t$ is a type I-factor and so $p(B(\mathcal{H})^{**}z_t)p$ = $pz_t B(\mathcal{H})^{**} pz_t$ is a type I-factor. Moreover

$(C^{oo}z_t p)' \cap z_t p B(\mathcal{H})^{**} z_t p = C^{oo} z_t p$; hence $C^{oo}z_t p$ is a masa in a type I-factor $z_t p B(\mathcal{H})^{**} z_t p$. Since $z_t p = P^{**}(z_t)p$,

$z_t p\, B(\mathcal{H})^{**} z_t p = p P^{**}(z_t) B(\mathcal{H})^{**} P^{**}(z_t) p$.

Moreover

$pP^{**}(pP^{**}(z_t) B(\mathcal{H})^{**} P^{**}(z_t) p)$

$= p P^{**}(z_t) C^{oo} P^{**}(z_t) = C^{oo} P^{**}(z_t) p$.

Therefore there exists a σ-continuous norm one projection $pP^{**}|pP^{**}(z_t)B(\mathcal{H})^{**}P^{**}(z_t)p$ of a type I-factor $pP^{**}(z_t)B(\mathcal{H})^{**}P^{**}(z_t)p$ onto

onto a masa $C^{**}p^{**}(z_t)p$, where $C^{**}p^{**}(z_t)p$
$\subset C^{**}p^{**}(z_t)$.

Now put $A = pp^{**}(z_t)B(H)^{**}p^{**}(z_t)p$ and
$B = C^{**}p^{**}(z_t)p$ and $pp^{**}|A = Q$; then
we have the identity of A ~~and B is~~ $pp^{**}(z_t)$

<u>Theorem 8.</u> B is an atomic masa of A.

Proof Q is a σ-continuous norm one projection of A onto B satisfying the following properties $pp^{**}(a^*a) \geq 0$ $(a \in A)$; $pp^{**}(p^{**}(z_t)p) = p^{**}(z_t)p$. $pp^{**}(b_1 a b_2) = b_1 pp^{**}(a) b_2$ for $b_1, b_2 \in B$ and $a \in A$.

For $f \in B_* \cap G(B)$, define $\hat{f}(a) = f(Q(a))$; then $\hat{f}$ is a normal state on B; hence there exists a positive trace class operator $\hat{f}$ in A_* such that $\hat{f}(a) = Tr(a\hat{f})$ $(a \in A)$;

$\hat{f}(uau^*) = f(Q(uau^*)) = f(uQ(a)u^*)$
$= f(Q(a)) = \hat{f}(a)$ $(a \in A, u \in B^u)$; hence
$Tr(a\hat{f}) = Tr(uau^*\hat{f}) = Tr(au^*\hat{f}u)$ $(a \in A)$

; hence $u^*\hat{f}u = \hat{f}$ $(u \in B^u)$. Since B is a masa, $\hat{f} \in B$; hence B is atomic ⌋

Therefore we have $z_t p = p^{**}(z_t) p = p^{**}(z_t)$ for $C^{**} p^{**}(z_t) p$ is atomic.

Hence $z_t p B(H)^{**} z_t p = p^{**}(z_t) B(H)^{**} p^{**}(z_t)$ and $p^{**}(z_t) \leqq p$ and $p^{**}(z_t) \leqq z_t$.

$$p^{**}(z_t) = \sum_{s(\varphi_s) \leqq z_t} s(\varphi_s), \text{ for } s(\varphi_s) \in C^{**}.$$

Theorem 9. For $u \in B(H)^u$ with $uCu^* = C$, $p^{**}(uzu^*) = u p^{**}(z) u^*$ for $z \in Z$ and so $p^{**}(z) = u p^{**}(z) u^*$.

Proof. $$\begin{array}{c} p^{**} \alpha_u(z) = p^{**}(uzu^*) = p^{**}(z) \\ \| \\ \alpha_u p^{**}(z) = u p^{**}(z) u^* \end{array} \qquad \lrcorner$$

$z_t - p^{**}(z_t) \in$ the center of D and $p^{**}(z_t - p^{**}(z_t)) = 0$, and so $z_t - p^{**}(z_t) \leqq g$.

Problem 2. Can one prove that there exist a point $t_0 \in \beta N$ such that $z_{t_0} - p^{**}(z_{t_0}) > 0$?

Theorem 10. For $z \in Z$, suppose that $B(H)^{**} z$ is a type I-factor and $p^{**}(z) \neq 0$; then there is a $t \in \beta N$ such that $z = z_t$.

Proof. Since $zp = p^{**}(z)p$, $zp \neq 0$.

$zpB(\mathcal{H})^{**}zp$ is a type I-factor and $(C^{cu}zp)' \cap zpB(\mathcal{H})^{**}zp = Dzp = C^{cu}zp$; hence $C^{cu}zp$ is a masa in $zpB(\mathcal{H})^{**}zp$.

Consider the mapping $zpP^{**} \mid zpB(\mathcal{H})^{**}zp$; it is a σ-continuous norm-one projection of $zpB(\mathcal{H})^{**}zp$ on $C^{cu}zp$ and so $C^{cu}zp$ = atomic (cf. the proof of Theorem 8) ; hence there exists a one-dimensional projection e_t such that $e_t \leq p^{**}(z)$ and so $z = z_t$ ⊥

Theorem 11. Assume the continuum Hypothesis. Then there exists a central projection z in $B(\mathcal{H})^{**}$ such that $B(\mathcal{H})^{**}z$ is a type I-factor and $p^{**}(z) = 0$. Hence $\mathfrak{g} \neq 0$.

Proof. Akemann-Weaver proved that under the assumption of the Continuum Hypothesis there exists a pure state on $B(\mathcal{H})$ whose restriction to any masa of $B(\mathcal{H})$ is not pure. Take such a state ψ and let z_ψ be the central support of ψ such that $B(\mathcal{H})^{**}z_\psi$ is unitary equivalent to $\{\pi_\psi, \mathcal{H}_\psi\}$ of $B(\mathcal{H})$.

If $p^{**}(z_y) \neq 0$, then by the above theorem, $\{\pi_y, \mathcal{H}_y\}$ is unitary equivalent to $\{\pi_{\tilde{\varphi}_t}, \mathcal{H}_{\tilde{\varphi}_t}\}$ for some $t \in \beta N$, a contradiction $\perp$

Remark The above theorem implies that the statement "$\mathfrak{z} \neq 0$" is consistent with ZFC.

Problem 3.. Is the statement of "$\mathfrak{z} = 0$" consistent with ZFC ?

Remark If the above problem is affirmative, then the statement "$\mathfrak{z} \neq 0$" is undecidable in ZFC.

Theorem 12. If "$\mathfrak{z} = 0$" is consistent with ZFC, then "the statement of "the problem of Kadison-Singer is affirmative for all $t \in \beta N$" is consistent with ZFC.

Proof. Suppose $\mathfrak{z} = 0$; then $\mathcal{D} = C^{\circ\circ}$; hence $C^{\circ\circ}$ is a masa in $B(\mathcal{H})^{**}$.
Therefore by Markov-Kakutani theorem, $V(a)^{\circ\circ} \cap \mathcal{D} = V(a)^{\circ\circ} \cap C^{\circ\circ} \neq \phi$; hence there is an element $y \in V(a)^{\circ\circ} \cap C^{\circ\circ}$; hence

Hence $\exists \{\sum \lambda_{j,\alpha} u_{j,\alpha}\, a\, u_{j,\alpha}^* : \lambda_{j,\alpha} \geq 0,\ \sum_j \lambda_{j,\alpha} = 1,$
$u_{j,\alpha} \in C^u\}$;

$$\sigma(B(\mathcal{H})^{**}, B(\mathcal{H})^*)\text{-}\lim \sum \lambda_{j,\alpha} u_{j,\alpha}\, a\, u_{j,\alpha}^* = y \ ;$$

hence $P^{**}(\sum \lambda_{j,\alpha} u_{j,\alpha}\, a\, u_{j,\alpha}^*) = P^{**}(a) = P(a) \to P^{**}(y)$.
On the other hand, $y \in C^{\circ\circ}$; hence $P^{**}(y) = y$

Hence $P(a) = y$ and so $\sum \lambda_{j,\alpha} u_{j,\alpha}\, a\, u_{j,\alpha}^* \to P(a)$
in $\sigma(B(\mathcal{H}), B(\mathcal{H})^{**})$. Hence $V(a) \cap C = \{P(a)\}$ ⌟

Problem 4. Can one prove that $C^{\circ\circ}$ is not a masa in $B(\mathcal{H})^{**}$ within ZFC ?

Remark It is known that $C + K(\mathcal{H}) / K(\mathcal{H})$
is a masa in $B(\mathcal{H}) / K(\mathcal{H})$, where
$K(\mathcal{H})$ = the C^* subalgebra of all compact linear operators on $\mathcal{H}$.

End

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