Phase transitions in a number-theoretic dynamical system

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This talk is about joint work with Marcelo Laca.

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In statistical physics, an important role is played by *equilibrium states*, which are in particular invariant under the time evolution.

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of *analytic elements* is a dense subalgebra of A. A state ϕ on A is a *KMS state at inverse temperature β* if

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- In a physical model we expect KMS states for most β.

The Cuntz algebra $\mathcal{O}_n$ is the universal algebra generated by $\{s_j : 1 \leq j \leq n\}$ satisfying $s_j^* s_j = 1 = \sum_{j=1}^n s_j s_j^*$. There is an action $\alpha : \mathbb{R} \to \operatorname{Aut} \mathcal{O}_n$ such that $\alpha_t(s_j) = e^{it} s_j$.

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$$s_\mu s_\nu^* := s_{\mu_1} \cdots s_{\mu_{|\mu|}} (s_{\nu_1} \cdots s_{\nu_{|\nu|}})^*,$$

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We have $\alpha_t(s_\mu s_\nu^*) = e^{it(|\mu| - |\nu|)} s_\mu s_\nu^*$, which makes sense for $t \in \mathbb{C}$, so the $s_\mu s_\nu^*$ are analytic elements.

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Lemma. ϕ is a KMS$_\beta$ state iff $\phi(s_\mu s_\nu^*) = \begin{cases} 0 & \text{if } \nu \neq \mu \\ e^{-\beta|\mu|} & \text{if } \nu = \mu. \end{cases}$

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If ϕ is a KMS$_\beta$ state on $\mathcal{O}_n$ then

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So $\mathcal{O}_n$ has at most one KMS$_\beta$ state, when $\beta = \log\log n$. Does it have one?

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Consider $\Sigma^* = \bigcup_{k\geq 0}\{1, \cdots, n\}^k$, and S_j on $\ell^2(\Sigma^*)$ defined by $S_j e_\mu = e_{j\mu}$, giving a representation π_S of $\mathcal{TO}_n$.

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$$\phi_\beta(a) = (1 - ne^{-\beta}) \sum_{\mu \in \Sigma^*} e^{-\beta|\mu|} (\pi_S(a) e_\mu \mid e_\mu)$$

defines a KMS$_\beta$ state on $\mathcal{T}\mathcal{O}_n$ for every $\beta > \log n$.

Any cancellative semigroup has a *Toeplitz representation* T on $\ell^2(P)$ such that $T_x e_y = e_{xy}$. The *Toeplitz algebra* of P is $\mathcal{T}(P) := C^*(T_x : x \in P) \subset B(\ell^2(P))$.

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Examples. $\mathcal{T}(\mathbb{N})$ is generated by the unilateral shift, and is the universal C^*-algebra generated by an isometry (Coburn 1967).

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$\mathcal{T}(\mathbb{N}^2)$ is generated by the isometries T_{e_1}, T_{e_2}, but is not universal for such pairs: T satisfies the extra relation $T_{e_1} T_{e_2}^* = T_{e_2}^* T_{e_1}$.

Consider a subsemigroup P of a group G which satisfies $P \cap P^{-1} = \{e\}$ and which generates G.

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For $x, y \in G$ we define $x \leq y \iff x^{-1}y \in P \iff y \in xP$, and then $\leq$ is a partial order on G.

Def. [Nica, 92]. (G, P) is *quasi-lattice ordered* if every pair $x, y \in G$ with a common upper bound in P has a least upper bound $x \vee y$ in P.

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(3) $\mathbb{F}_2$ is the free group with generators a, b, and $P = \langle a, b \rangle$. Then $x \leq y$ means that x is an initial segment of y, and the rest of y has no factors of a^{-1} or b^{-1}. Here $x \vee y = \infty$ often.

An isometric representation $V : P \to \mathrm{Isom}(H)$ is *Nica covariant* if

$$(V_x V_x^*)(V_y V_y^*) = \begin{cases} V_{x\vee y} V_{x\vee y}^* & \text{if } x \vee y < \infty \\ 0 & \text{if } x \vee y = \infty. \end{cases}$$

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Example. $V : \mathbb{N}^2 \to \mathrm{Isom}(H)$ is Nica covariant if and only if $V_{e_1} V_{e_2}^* = V_{e_2}^* V_{e_1}$.

Example. Nica-covariant representations of $(\mathbb{F}_2, \langle a, b\rangle)$ are given by pairs of isometries such that $(S_a S_a^*)(S_b S_b^*) = 0$, or equivalently $S_a S_a^* + S_b S_b^*$.

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The new work with Marcelo concerns the following semigroup recently studied by Cuntz:

Example. $\mathbb{N} \rtimes \mathbb{N}^\times$ with $(m, a)(n, b) = (m + an, ab)$.

Question 1. Is $(\mathbb{Q} \rtimes \mathbb{Q}_+^*, \mathbb{N} \rtimes \mathbb{N}^\times)$ quasi-lattice ordered?

We need to understand the partial order on $\mathbb{N} \rtimes \mathbb{N}^{\times}$. We have

$$\begin{aligned}(m, a) \leq (k, c) &\iff \exists\,(l, d) \in P \text{ with } (k, c) = (m + al, ad)\\ &\iff a|c \text{ and } k \in m + a\mathbb{N}.\end{aligned}$$

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Proposition. $(\mathbb{Q} \rtimes \mathbb{Q}_+^*, \mathbb{N} \rtimes \mathbb{N}^\times)$ *is quasi-lattice ordered, and for* $(m,a),(n,b) \in P = \mathbb{N} \rtimes \mathbb{N}^\times$ *we have*

$$(m,a) \vee (n,b) = \begin{cases} \infty & \text{unless } (a,b) \text{ divides } m-n, \\ (l,[a,b]) & \text{if it does,} \end{cases}$$

where $l = \min((m+a\mathbb{N}) \cap (n+b\mathbb{N}))$ *(which is non-empty if and only if* (a,b) *divides* $m-n$*).*

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In particular, $\mathcal{T}(\mathbb{N} \rtimes \mathbb{N}^{\times}) = \overline{\text{span}}\{T_{(m,a)} T_{(n,b)}^*\}$.

We set $S = T_{(1,1)}$ and $V_a = T_{(0,a)}$. Then

(a) $V_a V_b = V_b V_a$ for all a, b, and $V_a^* V_b = V_b V_a^*$ for $(a, b) = 1$,

(b) $V_a S = S^a V_a$,

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(a) says that V is a Nica-covariant representation of $\mathbb{N}^\times$; (b) says $T_{(0,p)} T_{(1,1)} = T_{(p,p)}$; (c) and (d) are Nica covariance for the pairs $x = (1, 1)$, $y = (0, p)$ and $x = (0, p)$, $y = (1, p)$.

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Corollary. There is a continuous action $\sigma : \mathbb{R} \to \operatorname{Aut} \mathcal{T}(\mathbb{N} \rtimes \mathbb{N}^\times)$ such that $\sigma_t(S) = S$ and $\sigma_t(V_a) = a^{it} V_a$.

We have

$$\sigma_t(S^m V_a V_b^* S^{*n}) = a^{it} b^{-it} S^m V_a V_b^* S^{*n} = e^{(\log a - \log b)it} S^m V_a V_b^* S^{*n},$$

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so all the spanning elements are analytic.The following lemma looks disarmingly easy:

Lemma. A state ϕ of $\mathcal{T}(\mathbb{N} \rtimes \mathbb{N}^\times)$ is a KMS$_\beta$ state if and only if

$$\phi(S^m V_a V_b^* S^{*n}) = \begin{cases} 0 \ \text{ if } a \neq b \text{ or } m \not\equiv n \pmod{a} \\ a^{-\beta} \phi(S^{a^{-1}(m-n)}) \ \text{ if } a = b,\, m - n \in a\mathbb{N} \\ a^{-\beta} \phi(S^{*a^{-1}(n-m)}) \ \text{ if } a = b,\, n - m \in a\mathbb{N}. \end{cases}$$

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To prove it, though, we have to show that the condition implies

$$\phi\big((S^m V_a V_b^* S^{*n})(S^k V_c V_d^* S^{*l})\big) = a^{it} b^{-it} \phi\big((S^k V_c V_d^* S^{*l})(S^m V_a V_b^* S^{*n})\big)$$

and this involves being able to compute least upper bounds in $\mathbb{N} \rtimes \mathbb{N}^\times$.

Theorem (Laca–R) *Consider the system* $(C^*(\mathbb{N} \rtimes \mathbb{N}^\times), \sigma)$ *described above. Then:*

For $\beta < 1$*, there are no KMS*$_\beta$ *states.*

For $1 \leq \beta \leq 2$*, there is a unique KMS*$_\beta$ *state.*

For $\beta > 2$*, the simplex of KMS*$_\beta$ *states is isomorphic to the simplex* $P(\mathbb{T})$ *of probability measures on the unit circle* $\mathbb{T}$*.*

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"We have spontaneous symmetry breaking as β increases through 2", but the circular symmetry which is being broken does not come from an action of $\mathbb{T}$ on $\mathcal{T}(\mathbb{N} \rtimes \mathbb{N}^\times)$.