Equivalence parameters and a canonical construction for GMRAs

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joint work with
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Ingredients

Let $\mathcal{H}$ be a Hilbert space equipped with the following operators:

π is a unitary representation of an abelian group Γ acting in $\mathcal{H}$.
δ is a unitary operator on $\mathcal{H}$.

We assume these operators are related by $\delta^{-1}\pi_\gamma\delta = \pi_{\alpha(\gamma)}$, where α is an isomorphism of Γ into itself whose image has finite index $N > 1$ in Γ. Write α^* for the dual isomorphism on $\widehat{\Gamma}$.

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The classic example of this is $\mathcal{H} = L^2(\mathbb{R}^d)$.
π is the representation of $\mathbb{Z}^d$ given by $\pi_n f(x) = f(x-n)$.
δ is dilation by an expansive (all eigenvalues with absolute value greater than 1) integer matrix A: $\delta f(x) = \sqrt{|\det A|} f(Ax)$.

$\alpha(n) = An$ and $\alpha^*(\omega) = A^*\omega$ for $\omega \in \mathbb{T}^d \equiv \mathbb{R}^d/\mathbb{Z}^d$.
We have $\delta^{-1}\pi_n\delta f(x) = f(x - An) = \pi_{An} f(x)$.

A collection $\{V_j\}_{-\infty}^{\infty}$ of closed subspaces of $\mathcal{H}$ is called a *Generalized Multiresolution Analysis (GMRA)* relative to π and δ if:

1. $V_j \subseteq V_{j+1}$ for all j.
2. $V_{j+1} = \delta(V_j)$ for all j.
3. $\cap V_j = \{0\}$, and $\cup V_j$ is dense in $\mathcal{H}$.
4. V_0 is invariant under the representation π.

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Classical examples for translation by $\mathbb{Z}$ and dilation by 2 in $L^2(\mathbb{R})$:

Haar MRA: $\phi = \chi_{[0,1]}$

Shannon MRA: $\widehat{V}_0 = L^2([-\frac{1}{2}, \frac{1}{2}])$

Journe GMRA:

$$\widehat{V}_0 = L^2([-\tfrac{8}{7}, -1] \cup [-\tfrac{4}{7}, -\tfrac{1}{2}) \cup [-\tfrac{2}{7}, \tfrac{2}{7}] \cup [\tfrac{1}{2}, \tfrac{4}{7}] \cup [1, \tfrac{8}{7}])$$

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GMRAs are a useful construct for applications such as image processing. Think of V_j as containing images at up to the j^{th} level of zoom.

GMRAs and wavelets:

If we write W_j for the orthogonal complement to V_j in V_{j+1}, then $\{\psi_k\}$ whose translates form an orthonormal basis (Parseval frame) for W_0 is an orthonormal (Parseval frame) wavelet.

Conversely, a wavelet gives rise to a GMRA by taking V_j to be the space spanned by dilates of translates of the wavelet with the power of dilation less than j.

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Stone's Theorem:

π restricted to V_0, as a unitary representation of an abelian group, is completely determined by a measure μ on $\widehat{\Gamma}$ and a multiplicity function $m : \widehat{\Gamma} \mapsto \{0, 1, 2, \cdots, \infty\}$, which essentially describes how many times each character occurs as a subrepresentation of $\pi|_{V_0}$. There is a unitary equivalence J between translation on V_0 and multiplication by characters on $\oplus L^2(\sigma_i)$, where $\sigma_i = \{\omega : m(\omega) \geq i\}$. We will restrict to the case where μ is Haar measure, and m is finite a.e.
In the MRA case for $\Gamma = \mathbb{Z}^d$, $m \equiv 1$, so $\oplus L^2(\sigma_i) = L^2(\mathbb{T}^d)$.

Think of J as a partial alternative Fourier transform.

Filters

The operator $J : V_0 \mapsto \bigoplus L^2(\sigma_i)$ interacts with translations according to $J\pi_\gamma f(\omega) = \omega(\gamma) Jf(\omega)$. What about the dilation?

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We have $\delta^{-1}\phi \in V_0$, so $J\delta^{-1}\phi = h$ for some function h on $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$. The function h will contain the exponentials corresponding to the the translates of ϕ that make up $\delta^{-1}\phi$.

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A Fourier argument shows that h must satisfy an orthogonality condition

$$\sum_{A^T(\zeta)=\omega} |h(\zeta)|^2 = |\det A| \quad a.e.\ \omega.$$

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Thus, $h = \frac{1}{\sqrt{2}}(1 + e_{-1})$, where $e_n(\omega) \equiv e^{2\pi i n\omega}$.
Orthogonality here says $|h(\frac{\omega}{2})|^2 + |h(\frac{\omega}{2} + \frac{1}{2})|^2 = 2$.

Filters: general case

We have $\delta^{-1} J^{-1} \chi_{\sigma_i} \in V_0$, so $J\delta^{-1} J^{-1} \chi_{\sigma_i} = \oplus_j h_{i,j}$, where the functions $h_{i,j} \in L^2(\sigma_j)$.

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For the Journe GMRA,

$$\widehat{V}_0 = L^2([-\frac{8}{7}, -1] \cup [-\frac{4}{7}, -\frac{1}{2}) \cup [-\frac{2}{7}, \frac{2}{7}] \cup [\frac{1}{2}, \frac{4}{7}] \cup [1, \frac{8}{7}]),$$

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$$\sigma_1 = [-\frac{1}{2}, -\frac{3}{7}] \cup [-\frac{2}{7}, \frac{2}{7}] \cup [\frac{3}{7}, \frac{1}{2}] \text{ and } \sigma_2 = [-\frac{1}{7}, \frac{1}{7}]$$

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$$H = \begin{pmatrix} \sqrt{2}\chi_{[-\frac{2}{7},-\frac{1}{4}]\cup[-\frac{1}{7},\frac{1}{7})\cup[\frac{1}{4},\frac{2}{7}]} & 0 \\ \sqrt{2}\chi_{[-\frac{1}{2},-\frac{3}{7}]\cup[\frac{3}{7},\frac{1}{2}]} & 0 \end{pmatrix}$$

The matrix $H = [h_{i,j}]$ must satisfy the orthogonality condition:

$$\sum_{\alpha^*(\zeta)=\omega} \sum_j h_{i,j}(\zeta)\overline{h_{i',j}(\zeta)} = N\delta_{i,i'}\chi_{\sigma_i}(\omega) \quad a.e\ \omega \in \widehat{\Gamma}.$$

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(For Journe, this says $|h_{1,1}(\frac{\omega}{2})|^2 + |h_{1,1}(\frac{\omega}{2} + \frac{1}{2})|^2 = 2\chi_{\sigma_1}$,

$$|h_{2,1}(\frac{\omega}{2})|^2 + |h_{2,1}(\frac{\omega}{2} + \frac{1}{2})|^2 = 2\chi_{\sigma_2}, \text{ and}$$

$$h_{1,1}(\tfrac{\omega}{2})\overline{h_{2,1}(\tfrac{\omega}{2})} + h_{1,1}(\tfrac{\omega}{2} + \tfrac{1}{2})\overline{h_{2,1}(\tfrac{\omega}{2} + \tfrac{1}{2})} = 0.)$$

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A matrix H that has $h_{i,j} \in L^2(\sigma_j)$ and satisfies this orthogonality condition is called a *filter*. The filter H determines how δ^{-1} acts in $J(V_0) = \oplus L^2(\sigma_i)$:

$$J\delta^{-1}J^{-1}f(\omega) = H^T(\omega)f(\alpha^*(\omega)).$$

To see how δ^{-1} acts on $W_0 = V_1 \ominus V_0$, we repeat our procedure for that space, which also must be invariant under Γ. Stone's theorem gives $\widetilde{J} : W_0 \mapsto \bigoplus L^2(\widetilde{\sigma}_k)$. We have

$$\widetilde{J}\pi_\gamma f(\omega) = \omega(\gamma)\widetilde{J}(f)(\omega) \text{ and}$$

$$J\delta^{-1}\widetilde{J}^{-1}\chi_{\widetilde{\sigma}_k} = \oplus_j g_{k,j},$$

where the matrix of functions $G = [g_{k,j}]$ is a filter. That is, $g_{k,j} \in L^2(\sigma_j)$, and

$$\sum_{\alpha^*(\zeta)=\omega} \sum_j g_{k,j}(\zeta)\overline{g_{k,j}(\zeta)} = N\delta_{k,k'}\chi_{\widetilde{\sigma}_i}(\alpha^*(\omega)).$$

The filters G and H are complementary in the sense that an additional orthogonality relation holds:

$$\sum_{\alpha^*(\zeta)=\omega} \sum_j g_{k,j}(\zeta)\overline{h_{i,j}(\zeta)} = 0.$$

Ruelle operators

J takes $\delta^{-1} : V_0 \to V_0$ to the the Ruelle operator $S_H : \bigoplus L^2(\sigma_i) \to \bigoplus L^2(\sigma_i)$:

$$J\delta^{-1}J^{-1}f(\omega) = S_H f(\omega) = H^T(\omega)f(\alpha^*(\omega)).$$

and $\widetilde{J}$ takes $\delta^{-1} : W_0 \to V_0$ to the Ruelle operator $S_G : \bigoplus L^2(\widetilde{\sigma}_k) \to \bigoplus L^2(\sigma_i)$:

$$J \circ \delta^{-1} \circ \widetilde{J}^{-1}(f)](\omega) = S_G f(\omega) = G^T(\omega)f(\alpha^*(\omega)).$$

The orthogonality conditions imply that the Ruelle operators S_H and S_G satisfy the following Cuntz-like relations:

1. $S_H^* S_H = I$, $S_G^* S_G = \widetilde{I}$,
2. $S_H^* S_G = 0$, and
3. $S_H S_H^* + S_G S_G^* = I$,

where I is the identity operator on $\bigoplus_i L^2(\sigma_i)$ and $\widetilde{I}$ is the identity operator on $\bigoplus_k L^2(\widetilde{\sigma}_k)$.

Building GMRAs from filters

Mallat, Meyer and Daubechies pioneered the use of filters to build MRAs and wavelets. The idea was to use an iteration of a Fourier transform version of our definition of h:

$$\widehat{\phi}(\omega) = \frac{1}{\sqrt{|\det A|}} h(\frac{\omega}{2}) \widehat{\phi}(\frac{\omega}{2})$$

to build

$$\widehat{\phi} = \Pi_{j=1}^{\infty} \frac{1}{\sqrt{|\det A|}} h((A^T)^{-j}\omega).$$

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A Fourier transform version of our definition of g could then be used to build

$$\widehat{\psi}(\omega) = \frac{1}{\sqrt{|\det A|}} g(\frac{\omega}{2})\widehat{\phi}(\frac{\omega}{2})$$

The GMRA results from taking V_0 and W_0 to be spanned by translates of ϕ and ψ respectively.

These methods required a potential filter function h to satisfy:

1. the orthogonality condition $\sum_{A^T(\zeta)=\omega} |h(\zeta)|^2 = \det A \; a.e \; \omega$
2. a "low-pass" condition: that h take on values close to $\sqrt{|\det A|}$ near the origin. This ensures convergence of the infinite product
3. a "Cohen" condition: that h not vanish is some neighborhood of the origin. This ensures L^2 convergence so that the translates of ϕ andψ would be orthonormal.

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Lawton/Bratelli/Jorgensen removed the Cohen condition (cost: replacing orthonormal bases with frames.)
Baggett/Courter/Jorgensen/M/Packer extended this work to GMRAs (replaced h with H=matrix filter).

The low-pass condition

Some version of the low-pass condition would be required to make the infinite product converge. Thus, eliminating low-pass means finding another method of building the GMRA out of the filter. This in turn leads to building GMRAs in spaces other than $L^2(\mathbb{R}^d)$.

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Dutkay/Jorgensen (2004) produced an MRA (and wavelet) based on the Cantor set $\mathcal{C}$ in $L^2(\mathcal{R})$, where $\mathcal{R}$ is the set of reals with only finitely many 1's in their ternary expansions. V_0 is spanned by translates of $\chi_{\mathcal{C}}$, and $h = \frac{1}{\sqrt{2}}(1 + e_2)$ (where $e_n(x) = e^{2\pi i n x}$). This filter does not satisfy low-pass, as its value at the origin is $\sqrt{2}$ rather than $\sqrt{3}$.

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D'Andrea/M/Packer worked out a similar construction for the Sierpinski triangle.

Direct limit constructions of GMRAs

Larsen/Raeburn, later with Baggett/M/Packer/Ramsay, as well as Dutkay/Jorgensen, have realized GMRAs as direct limits.

This method builds GMRAs via the Ruelle operator S_H that represents δ^{-1} on $\mathcal{K} \equiv \bigoplus L^2(\sigma_i)$. If S_H is a pure isometry ($\cap_{n=1}^{\infty} S_H^n \mathcal{K} = 0$), then the Hilbert-space direct limit, $\varinjlim (K, S_H)$ is naturally equipped with a GMRA structure.

The above authors have identified these direct limits with concrete realizations in the case of some classical and fractal filters.

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Questions

1. The ingredients of a GMRA appear to be a multiplicity function and filters. What criteria do these need to satisfy in order to yield a GMRA? (Evidently less restrictive if we do not require the GMRA to be in $L^2(\mathbb{R})$.)

2. Is there a universal concrete construction technique for a GMRA given these ingredients?

Restrictions on the multiplicity function m

If a nonzero function $m : \widehat{\Gamma} \mapsto \{0, 1, 2, \cdots\}$ is a multiplicity function for a GMRA, m must satisfy the *consistency inequality*:

$$m(\omega) \leq \sum_{\alpha^*(\zeta)=\omega} m(\zeta).$$

In this case we can write

$$\widetilde{m}(\omega) = \sum_{\alpha^*(\zeta)=\omega} m(\zeta) - m(\omega),$$

and use $\widetilde{m}$ as the multiplicity function on W_0.

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Bownik/Rzeszotnik/Speegle and Baggett/M showed that an additional technical condition (related to dilates of translates of the support of m) is required for m to be a multiplicity function for a GMRA in $L^2(\mathbb{R}^d)$.

Restrictions on the filter H

For the filter H to be associated with a GMRA, we need the corresponding Ruelle operator S_H to be a pure isometry.

For H a 1×1 matrix, S_H is a pure isometry if $|H(\omega)| \neq 1$ on a set of positive measure in $\widehat{\Gamma}$.

For general H a more technical result shows essentially that S_H is a pure isometry if there exists a set of positive measure where $H = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, with A expansive and B, C, D small.

This is significantly weaker than the traditional low-pass assumption that h be close to the $\sqrt{|detA|}$ near 0, or the matrix low-pass conditions that require H have an upper left corner close to $\sqrt{|detA|} \times I$ near the origin. For example, S_h will be a pure isometry if we use a traditional high pass filter for h.

A canonical construction for GMRAs

Theorem (BFMP): Given a group Γ with an isomorphism α. Suppose $m : \widehat{\Gamma} \to 0, 1, 2, \cdots$ is a Borel function that satisfies the consistency inequality, and that $H = [h_{i,j}]$ is a $m(\alpha^*(\omega)) \times m(\omega)$ matrix valued function on $\widehat{\Gamma}$ satisfying:

(1) $h_{i,j}$ supported on σ_j

(2) orthogonality: $\sum_{\alpha^*(\zeta)=\omega} \sum_j h_{i,j}(\zeta)\overline{h_{i',j}(\zeta)} = N\delta_{i,i'}\chi_{\sigma_i}(\omega)$

(3)S_H is a pure isometry on $\bigoplus L^2(\sigma_i)$.

Then, there exists a complementary filter $G = [g_{k,j}]$ with $g_{k,j}$ supported on σ_j, which satisfies orthogonality with respect to $\widetilde{m}(\omega) = \sum_{\alpha^*(\zeta)=\omega} m(\zeta) - m(\omega)$, and is orthogonal to H.

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Theorem (BFMP): Given a group Γ with an isomorphism α. Suppose $m : \widehat{\Gamma} \to 0, 1, 2, \cdots$ is a Borel function that satisfies the consistency inequality, and that $H = [h_{i,j}]$ is a $m(\alpha^*(\omega)) \times m(\omega)$ matrix valued function on $\widehat{\Gamma}$ satisfying:

(1) $h_{i,j}$ supported on σ_j

(2) orthogonality: $\sum_{\alpha^*(\zeta)=\omega} \sum_j h_{i,j}(\zeta)\overline{h_{i',j}(\zeta)} = N\delta_{i,i'}\chi_{\sigma_i}(\omega)$

(3)S_H is a pure isometry on $\bigoplus L^2(\sigma_i)$.

Then, there exists a complementary filter $G = [g_{k,j}]$ with $g_{k,j}$ supported on σ_j, which satisfies orthogonality with respect to $\widetilde{m}(\omega) = \sum_{\alpha^*(\zeta)=\omega} m(\zeta) - m(\omega)$, and is orthogonal to H.

For each choice of G, there is a canonical GMRA $\{V_j^{m,H,G}\}$ in

$$\mathcal{H} = \left(\bigoplus_i L^2(\sigma_i)\right) \oplus \left(\bigoplus_k L^2(\widetilde{\sigma}_k)\right) \oplus \bigoplus_{j=1}^{\infty} \mathcal{D}^j \left(\bigoplus_k L^2(\widetilde{\sigma}_k)\right)$$

$$= V_0^{m,H,G} \oplus W_0^{m,H,G} \oplus \bigoplus_{j=1}^{\infty} W_j^{m,H,G}$$

$$\begin{aligned}\mathcal{H} &= \left(\bigoplus_i L^2(\sigma_i)\right) \oplus \left(\bigoplus_k L^2(\widetilde{\sigma}_k)\right) \oplus \bigoplus_{j=1}^{\infty} \mathcal{D}^j \left(\bigoplus_k L^2(\widetilde{\sigma}_k)\right) \\ &= V_0^{m,H,G} \oplus W_0^{m,H,G} \oplus \bigoplus_{j=1}^{\infty} W_j^{m,H,G}\end{aligned}$$

The group Γ acts on $\mathcal{H}$ by:

$$\pi_\gamma(\oplus f_l)(\omega) = \omega(\gamma)(\oplus f_l),$$

and has multiplicity function m. The unitary operator

$$\delta^{-1} = S_H \oplus S_G \oplus \mathcal{D}^{-1}$$

interacts with π_γ by $\delta^{-1}\pi_\gamma\delta = \pi_{\alpha(\gamma)}$.

In general, $\mathcal{D}$ is defined in terms of a cross section for the map α^*. For $\Gamma = \mathbb{Z}^d$, $\alpha(n) = An$ (where A is an expansive matrix), we embed $\widehat{\Gamma} = \mathbb{T}^d$ in $\mathbb{R}^d$, and define

$$\mathcal{D}^j(\oplus_k f_k(\omega)) = \bigoplus_k \frac{1}{\sqrt{|\det A|}^j} f_k((A^*)^{-j}\omega).$$

Example 1

Take $\Gamma = \mathbb{Z}$ and $\alpha(n) = 2n$. Let $m \equiv 1$ and

$$h = \chi_{[-\frac{1}{4},\frac{1}{4}]} \in L^2(\mathbb{T}) \equiv L^2([-\frac{1}{2},\frac{1}{2}]).$$

We check that $|h(\frac{x}{2})|^2 + |h(\frac{x}{2}+\frac{1}{2})|^2 = 2$. We easily find a complementary filter $g = \sqrt{2}\chi_{\pm[\frac{1}{4},\frac{1}{2}]}$.

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The canonical Hilbert space is $L^2(\mathbb{T}) \oplus L^2(\mathbb{T}) \oplus (\bigoplus_{j=1}^{\infty} L^2(2^j\mathbb{T}))$,

with $\pi_n(\oplus f_l) = e_n(\oplus f_l)$ (where $e_n(x) = e^{2\pi inx}$), and

$$\delta^{-1}(f_1 \oplus f_2 \oplus (\oplus_{j=1}^{\infty} f_{3,j}) = \sqrt{2}\left(\chi_{[-\frac{1}{4},\frac{1}{4}]}(\omega)f_1(2\omega) + \chi_{\pm[\frac{1}{4},\frac{1}{2}]}(\omega)f_2(2\omega)\right) \oplus \sqrt{2}f_{3,1}(2\omega) \oplus \left(\oplus_{j=2}^{\infty}\sqrt{2}^j f_{3,j}(2\omega)\right).$$

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$$\sqrt{2} f_{3,1}(2\omega) \oplus \left(\oplus_{j=2}^{\infty} \sqrt{2}^{j} f_{3,j}(2\omega)\right).$$

By mapping $W_j^{m,H,G} \mapsto L^2(\pm 2^j[\frac{1}{2},1])$, we can map this canonical GMRA to the Fourier transform of the Shannon GMRA.

Example 2: low-pass=high-pass

In the same setting, take $h = \sqrt{2}\chi_{\pm[\frac{1}{4},\frac{1}{2}]}$, $g = \sqrt{2}\chi_{[-\frac{1}{4},\frac{1}{4}]}$

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So, here,

$$\delta^{-1} : V_0^{m,H,G} = L^2([-\frac{1}{2},\frac{1}{2}]) \mapsto L^2(\pm[\frac{1}{4},\frac{1}{2}]) \mapsto L^2(\pm[\frac{3}{8},\frac{1}{2}]) \mapsto \cdots$$

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Because $h = 0$ on an interval around 0, this GMRA cannot be embedded in $L^2(\mathbb{R})$.

Example 3: Using the Journe m and H

Recall the Journe multiplicity function and filter given by

$$\sigma_1 = [-\frac{1}{2}, -\frac{3}{7}] \cup [-\frac{2}{7}, \frac{2}{7}] \cup [\frac{3}{7}, \frac{1}{2}] \text{ and } \sigma_2 = [-\frac{1}{7}, \frac{1}{7}] \text{ and}$$

$$H = \begin{pmatrix} \sqrt{2}\chi_{[-\frac{2}{7},-\frac{1}{4}]\cup[-\frac{1}{7},\frac{1}{7})\cup[\frac{1}{4},\frac{2}{7}]} & 0 \\ \sqrt{2}\chi_{[-\frac{1}{2},-\frac{3}{7}]\cup[\frac{3}{7},\frac{1}{2}]} & 0 \end{pmatrix}$$

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Since we know the Journe GMRA has an associated single orthonormal wavelet, $\widetilde{m} \equiv 1$. We can take G to be

$$G = \begin{pmatrix} \sqrt{2}\chi_{[-\frac{1}{4},-\frac{1}{7}]\cup[\frac{1}{7},\frac{1}{4}]} & \sqrt{2}\chi_{[-\frac{1}{7},\frac{1}{7}]} \end{pmatrix}$$

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Here $V_0^{m,H,G} = L^2(\sigma_1) \oplus L^2(\sigma_2)$, and $W_j^{m,H,G} = L^2(2^j\mathbb{T})$, $j \geq 0$.

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This canonical GMRA can be mapped to the usual Journe GMRA by integrally translating σ_1 and σ_2 to the scaling set, and $[-\frac{1}{2}, \frac{1}{2}]$ to the wavelet set.

Example 4: An alternative Journe GMRA

Using again the Journe multiplicity function, let

$$H = \begin{pmatrix} \sqrt{2}\chi_{[-\frac{2}{7},-\frac{1}{4}]\cup[-\frac{1}{7},\frac{1}{7})\cup[\frac{1}{4},\frac{2}{7}]} & 0 \\ 0 & \sqrt{2}\chi_{[-\frac{1}{14},\frac{1}{14}]} \end{pmatrix}$$

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We have

$$V_0^{m,H,G} = L^2([-\frac{2}{7},\frac{2}{7}]\cup\pm[\frac{3}{7},\frac{1}{2}])\oplus L^2([-\frac{1}{7},\frac{1}{7}))$$

$$V_{-1}^{m,H,G} = L^2([-\frac{1}{7},\frac{1}{7}]\cup\pm[\frac{1}{4},\frac{2}{7}])\oplus L^2([-\frac{1}{14},\frac{1}{14}])\cdots$$

Because all the V_{-j} have overlap between the direct summands, we cannot map to $L^2(\mathbb{R})$ in a way that eliminates overlap. Thus, this GMRA cannot exist in $L^2(\mathbb{R})$. If we use these filters, we get a GMRA with a degenerate multiplicity function that takes only the values 0 and 1.

Equivalence of GMRAs

We say that a GMRAs $\{V_j\}$ in the Hilbert space $\mathcal{H}$ with the representation π and dilation δ is *equivalent* to $\{V_j'\}$ in $\mathcal{H}'$ with π' and δ' if there exists a unitary operator $U : \mathcal{H} \to \mathcal{H}'$ that satisfies:

$U(V_j) = V_j'$ for all j.
$U \circ \pi_\gamma = \pi'_\gamma \circ U$ for all $\gamma \in \Gamma$.
$U \circ \delta = \delta' \circ U$.

In $L^2(\mathbb{R}^d)$, the Fourier transform gives an equivalence between any GMRA $\{V_j\}$ and $\{\widehat{V}_j\}$.

If an operator U gives an equivalence between $\{V_j\}$ and $\{V_j'\}$, two GMRA's for dilation by A and translation by $\mathbb{Z}^d$ in $L^2(\mathbb{R}^d)$, then $\widehat{U}$ is multiplication by a function u with absolute value 1, and such that $u(A^{*j}\omega) = u(\omega)$ for all integers j. Thus equivalence between GMRAs for the same dilation in $L^2(\mathbb{R}^d)$ generalizes the notion of different MSF wavelets attached to the same wavelet set.

Two GMRA's $\{V_j\}$ and $\{V'_j\}$ are equivalent if and only if there exist unitary operators $P : V_0 \mapsto V'_0$ and $Q : W_0 \mapsto W'_0$ that intertwine π_γ with π'_γ and δ^{-1} with δ'^{-1}.

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Using this, we see that any GMRA $\{V_j\}$ is equivalent to the canonical GMRA $\{V_j^{m,H,G}\}$, where m is its multiplicity function, $\oplus_j h_{i,j} = \delta^{-1} J^{-1} \chi_{\sigma_i}$ and $\oplus_j g_{k,j} = \delta^{-1} \widetilde{J}^{-1} \chi_{\widetilde{\sigma}_k}$.

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Canonical GMRAs for familiar examples

Any MRA for dilation by 2 in $L^2(\mathbb{R})$ that is associated with a single wavelet has canonical Hilbert space

$$L^2(\mathbb{T}) \oplus L^2(\mathbb{T}) \oplus (\bigoplus_{j=1}^{\infty} L^2(2^j\mathbb{T})),$$

$$\text{with } \pi_n(\oplus f_l) = e_n(\oplus f_l) \text{ (where } e_n(x) = e^{2\pi inx}\text{), and}$$

$$\delta^{-1}(f_1 \oplus f_2 \oplus (\oplus_{j=1}^{\infty} f_{3,j}) = (h(\omega)f_1(2\omega) + g(\omega)f_2(2\omega)) \oplus \sqrt{2}f_{3,1}(2\omega) \oplus \left(\oplus_{j=2}^{\infty} \sqrt{2}^j f_{3,j}(2\omega)\right).$$

We already saw that for the Shannon MRA, with $\widehat{V}_0 = L^2([-\frac{1}{2}, \frac{1}{2}])$, $h = \sqrt{2}\chi_{[-\frac{1}{4},\frac{1}{4}]}$ and $g = \sqrt{2}\chi_{\pm[\frac{1}{4},\frac{1}{2}]}$ this canonical GMRA is close to the Fourier transform of Shannon. For the Haar MRA, with V_0 spanned by translates of $\chi_{[0,1]}$, $h = \frac{1}{\sqrt{2}}(1 + e_{-1})$, $g = \frac{1}{\sqrt{2}}(e_{-1} - 1)$, this is not the case.

Dilation by 3

The MRA Haar 2-wavelet for dilation by 3 in $L^2(\mathbb{R})$ has canonical Hilbert space

$$L^2(\mathbb{T}) \oplus \left(L^2(\mathbb{T}) \oplus L^2(\mathbb{T})\right) \oplus \left(\bigoplus_{j=1}^{\infty} L^2(3^j\mathbb{T}) \oplus L^2(3^j\mathbb{T})\right).$$

The canonical $\delta^{-1} = S_h \oplus (S_{g_1} \oplus S_{g_2}) \oplus \left(\bigoplus_{j=1}^{\infty} \mathcal{D}^{-j}\right)$, where

$$h = \frac{1}{\sqrt{3}}(1+e_1+e_2), \quad g_1 = \frac{1}{\sqrt{2}}(e_1-e_2) \text{ and } g_2 = \frac{1}{\sqrt{6}}(-2+e_1+e_2),$$

and $\mathcal{D}^{-j}(f_1 \oplus f_2)(\omega)) = \sqrt{3}^j (f_1 \oplus f_2)(3^j\omega)$.

The Cantor set MRA has the same canonical GMRA except with

$$h = \frac{1}{\sqrt{2}}(1+e_2), \quad g_1 = e_1 \text{ and } g_2 = \frac{1}{\sqrt{2}}(1-e_2).$$

The latter cannot be realized in $L^2(\mathbb{R})$.

Equivalence between canonical GMRAs

Two canonical GMRA's $\{V_j^{m,H,G}\}$ and $\{V'^{m',H',G'}_j\}$ are equivalent if and only if $m = m'$ and there exist unitary matrix-valued functions A and B such that

$$H(\omega)A(\omega) = A(\alpha^*(\omega))H'(\omega) \text{ and}$$

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For two canonical MRA's, equivalence implies that $|h| = |h'|$ and $|g| = |g'|$. However, this is not sufficient, as a simple Fourier argument shows that there exists no function a of absolute value 1 such that $h(\omega)a(\omega) = -a(2\omega)h(\omega)$.

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Determining which h's are equivalent requires determining which functions are coboundaries. The fact that $e_n h$ is equivalent to h corresponds to the fact that an integer translate of a scaling function ϕ gives the same MRA.