Co-universal algebras for product systems

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Outline

- Cuntz-Pimsner algebras
- Product systems of Hilbert bimodules and the Toeplitz algebra $\mathcal{T}_{\mathrm{cov}}(X)$
- Sims-Yeend's $\mathcal{NO}_X$
- The core of $\mathcal{T}_{\mathrm{cov}}(X)$
- The C^*-algebra $\mathcal{NO}_X^{\mathrm{r}}$ and its co-universal property
- The gauge-invariant uniqueness property
- Applications

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Important feature

Katsura'a modified definition of $\mathcal{O}_X$ ensures that the canonical covariant rep. $X \to \mathcal{O}_X$ is injective and $\mathcal{O}_X$ satisfies a gauge-invariant uniqueness property with respect to the canonical gauge-action of $\mathbb{T}$.

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1. $\psi_p := \psi|_{X_p} \to B$ linear, $\forall p$; ψ_e homomorphism;
2. ψ is multiplicative;
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The *Toeplitz algebra* $\mathcal{T}_X$ of X is the universal C^*-algebra for Toeplitz reps. of X; let $i : X \to \mathcal{T}_X$ be the universal Toeplitz rep.

Compactly aligned product systems

Quasi-lattice ordered group (Nica 1992). Let G a discrete group, P subsemigroup with $P \cap P^{-1} = \{e\}$, and define $g \leq h \iff g^{-1}h \in P$ (partial order). Then (G, P) is quasi-lattice ordered if for all $p, q \in G$ with common upper bound in P there is a lub $p \vee q$ in P. Write $p \vee q < \infty$ when p, q have a common upper bound, else write $p \vee q = \infty$.

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Notation: for $p \leq t$, $p \neq e$ let $\iota_p^t : \mathcal{L}(X_p) \to \mathcal{L}(X_t)$ be defined by $\iota_p^t(S)(xy) = (Sx)y$ for $S \in \mathcal{L}(X_p)$, $x \in X_p$ and $t \in X_{p^{-1}t}$; let $\iota_e^t : \mathcal{K}(X_e) \to \mathcal{L}(X_t)$ be the left action ϕ_t.

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A product system X over (G, P) is compactly aligned (Fowler 2002) if

$$\iota_p^{p\vee q}(S)\iota_q^{p\vee q}(T) \in \mathcal{K}(X_{p\vee q}),$$

for all $S \in \mathcal{K}(X_p)$, $T \in \mathcal{K}(X_q)$, $p \vee q < \infty$, $p, q \in P$. (Fowler assumes essential bimodules, we don't.)

Nica covariant representations of product systems

Recall (Pimsner): when ψ_p is a Toeplitz representation, there is a homomorphism $\psi^{(p)} \colon \mathcal{K}(X_p) \to B$ s.t. $\psi^{(p)}(x \otimes y^*) = \psi_p(x)\psi_p(y)^*$ for all $x, y \in X_p$, $p \in P$.

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$$\psi^{(p)}(S)\psi^{(q)}(T) = \begin{cases} \psi^{(p\vee q)}\left(\iota_p^{p\vee q}(S)\iota_q^{p\vee q}(T)\right) & \text{if } p \vee q < \infty \\ 0 & \text{otherwise} \end{cases}$$

for all $S \in \mathcal{K}(X_p)$ and $T \in \mathcal{K}(X_q)$.

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for $p, q \in P$, $S \in \mathcal{K}(X_p)$, $T \in \mathcal{K}(X_q)$, with $\iota_p^{p\vee q}(S)\iota_q^{p\vee q}(T) = 0$ if $p \vee q = \infty$. Define $\mathcal{T}_{\mathrm{cov}}(X) := \mathcal{T}_X/\mathcal{I}$. (Note: this def. bypasses Fowler's assumption of all X_p essential.)

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Universal property of $(\mathcal{T}_{\rm cov}(X), i_X)$: given $\psi : X \to B$ Nica covariant rep., there is a $*$-homom. $\psi_* : \mathcal{T}_{\rm cov}(X) \to B$ s.t. $\psi_* \circ i_X = \psi$.

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$$\mathcal{T}_{\mathrm{cov}}(X) = \overline{\operatorname{span}}\,\{\, i_X(x)i_X(y)^* \mid x, y \in X \,\}.$$

Sims-Yeends' Cuntz-Nica-Pimsner algebra $\mathcal{NO}_X$

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For X compactly aligned product system over (G, P) quasi-lattice ordered, Sims and Yeend introduced a new Cuntz-Pimsner condition under the name Cuntz-Nica-Pimsner covariance.

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Definition

(Sims-Yeend (2007)) $\mathcal{NO}_X$ is the universal C^*-algebra generated by a Cuntz-Nica-Pimsner covariant rep. of X.

Towards Cuntz-Nica-Pimsner covariance

Let (G, P) be quasi-lattice ordered and X compactly aligned product system over P. Denote $\phi_p : A \to \mathcal{L}(X_p)$ for $p \in P$ (left actions). Let $I_e = A$ and $I_r := \cap_{e<s\leq r} \ker(\phi_s)$.

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Cuntz-Nica-Pimsner covariant representations

Definition

(Sims-Yeend (2007)) A Nica covariant representation $\psi : X \to B$ *is Cuntz-Nica-Pimsner covariant (CNP-covariant) if*

$$\left.\begin{array}{l} \forall F \subset P \text{ finite} \\ \forall T_p \in \mathcal{K}(X_p), p \in F \\ \sum_{p\in F} \tilde{\iota}_p^q(T_p) = 0 \text{ for large } q \end{array}\right| \Rightarrow \quad \sum_{p\in F} \psi^{(p)}(T_p) = 0_B.$$

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Theorem

(Sims-Yeend 2007) If all left actions $\tilde{\phi}_q$ for $q \in P$ are injective, the canonical CNP-representation $j_X : X \to \mathcal{NO}_X$ is injective.

The gauge-coaction

Proposition

Let (G, P) *quasi-lattice ordered and* X *compactly aligned product system over* P. *There is a (full) coaction* δ *of* G *on* $\mathcal{T}_{\mathrm{cov}}(X)$ *s. t.* $\delta(i_X(x)) = i_X(x) \otimes i_G(d(x)), \forall x \in X$.

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Proposition

There is a coaction ν of G on $\mathcal{N}\mathcal{O}_X$ making the diagram commute:

$$\begin{array}{ccc} \mathcal{T}_{\rm cov}(X) & \xrightarrow{q_{\rm CNP}} & \mathcal{N}\mathcal{O}_X \\ \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \nu} \\ \mathcal{T}_{\rm cov}(X) \otimes C^*(G) & \xrightarrow{q_{CNP} \otimes \mathrm{id}} & \mathcal{N}\mathcal{O}_X \otimes C^*(G) \end{array}$$

Gauge-invariant uniqueness property (part I)

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Traditional line of proving GIUP: first, assuming injectivity of $\Pi\psi|_{j_X(A)}$, show that the homomorphism $\Pi\psi$ given by the universal property of $\mathcal{NO}_X$ is injective on the fixed-point algebra under the gauge-coaction.

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Traditional line of proving GIUP: first, assuming injectivity of $\Pi\psi|_{j_X(A)}$, show that the homomorphism $\Pi\psi$ given by the universal property of $\mathcal{NO}_X$ is injective on the fixed-point algebra under the gauge-coaction. Then try to use intertwining conditional expectations (assuming B admits a coaction compatible with ν) to show $\Pi\psi$ is injective.

Injectivity on the core

The core of $\mathcal{T}_{\mathrm{cov}}(X)$ is

$$\mathcal{F} := \{i_X(x)i_X(y)^* \mid x, y \in X, d(x) = d(y)\} = (\mathcal{T}_{\mathrm{cov}}(X))^\delta.$$

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Theorem

(Carlsen-L-Sims-Vittadello) Let (G, P) be quasi-lattice ordered group and X a compactly aligned product system over P of right-Hilbert A–A bimodules. Assume either that the left actions ϕ_p on the fibres are all injective, or that P is directed and all $\tilde{\phi}_q$ are injective. Let $\psi\colon X \to B$ be a CNP-covariant rep. of X in a C^*-algebra B. Then the induced homomorphism $\Pi\psi : \mathcal{NO}_X \to B$ is injective on $q_{\mathrm{CNP}}(\mathcal{F})$ if and only if ψ is injective as a Toeplitz representation.

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So far so good, but...

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Claim

A quotient of $\mathcal{NO}_X$ will be helpful in understanding when the gauge-invariant uniqueness property holds.

Some notation

Recall that δ on $\mathcal{T}_{\mathrm{cov}}(X)$ satisfies $\delta(i_X(x)) = i_X(x) \otimes i_G(d(x))$ for all $x \in X$.

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Recall that $j_X : X \to \mathcal{NO}_X$ is injective if all left actions $\tilde{\phi}_q$ are injective (Sims-Yeend). This hypothesis holds when either all left actions ϕ_p on X_p are injective, or every bounded subset of P has a maximal element.

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P is directed if $p \vee q < \infty$ for all $p, q \in P$ (Nica).

Main theorem (Carlsen-L-Sims-Vittadello)

Hypotheses: Let (G, P) be quasi-lattice ordered and X a compactly aligned product system over P of Hilbert A–A bimodules. Suppose either that the left action $\phi_{\boldsymbol{p}}$ on each fibre is injective, or that P is directed and all $\tilde{\phi}_{\boldsymbol{q}}$ are injective.

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- **Existence.** There exists $(\mathcal{NO}_X^r, j_X^r, \nu^n)$ s.t. $j_X^r : X \to \mathcal{NO}_X^r$ is an injective CNP-covariant rep. which is gauge-compatible via the normal coaction ν^n of G on $\mathcal{NO}_X^r$.

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- **Co-universal property.** If $\psi : X \to B$ is an injective gauge-compatible Nica covariant rep. whose image generates B then there is a surjective $*$-homomorphism $\phi : B \to \mathcal{NO}_X^{\mathrm{r}}$ s.t. $\phi \circ \psi = j_X^{\mathrm{r}}$.

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- **Uniqueness.** If (C, ρ, γ) satisfies the same conditions, there is an isomorphism $\phi\colon C \to \mathcal{NO}_X^{\mathrm{r}}$ s.t. $j_X^{\mathrm{r}} = \phi \circ \rho$.

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So $\mathcal{NO}_X^{\mathrm{r}}$ defined by its co-universal property involving only Nica covariant reps. is more "accessible" an object than $\mathcal{NO}_X$ given by its universal property involving the difficult to check CNP-covariance.

An aside: coactions and Fell bundles

If δ is a coaction of a discrete group G on a C^*-algebra A, let $A_g^\delta := \{\, a \in A \mid \delta(a) = a \otimes i_G(g) \,\}$ for $g \in G$. The disjoint union of $A_g^\delta \times \{g\}$ for $g \in G$ forms a Fell bundle $\mathcal{A}$ over G (Quigg 1996).

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Equivalent conditions

Thus, from $(\mathcal{NO}_X, G, \nu)$ we form a Fell bundle $\mathcal{N}$, we let $\mathcal{NO}_X^{\mathrm{r}}$ be its reduced cross sectional algebra, and we let ν^{n} be the normal coaction on $\mathcal{NO}_X^{\mathrm{r}}$ obtained as the normalisation of ν.

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4. $\lambda_{\mathcal{N}} : \mathcal{N}\mathcal{O}_X \to \mathcal{N}\mathcal{O}_X^r$ *is an isomorphism.*

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$$\pi(g) \vee \pi(h) = \pi(g \vee h) \text{ and } \pi(g) = \pi(h) \implies g = h.$$

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4. *The Fell bundle $\mathcal{B} = (\mathcal{T}_{\mathrm{cov}}(X)^{\delta}_{g} \times \{g\})_{g \in G}$ has the approximation property.*

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4. *The Fell bundle $\mathcal{B} = (\mathcal{T}_{\mathrm{cov}}(X)^{\delta}_{g} \times \{g\})_{g \in G}$ has the approximation property.*
5. *The Fell bundle $\mathcal{N} = \big((\mathcal{NO}_X)^{\nu}_{g} \times \{g\}\big)_{g \in G}$ has the approximation property.*

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Corollary

$A \times_\alpha G \cong \mathcal{NO}_X.$

Yeend's topological higher-rank graphs (THRG)

For $k \in \mathbb{N}$, a *topological k-graph* is a pair (Λ, d) consisting of:
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No gauge-invariant uniqueness thm. is established for $C^*(\mathcal{G}_\Lambda)$.

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Given a compactly aligned topological higher rank graph Λ, we construct a compactly aligned (involves non-trivial arguments!) product system X over $\mathbb{N}^k$ with fibers X_n as completions of $C_c(\Lambda^n)$.

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The co-universal property implies that $\mathcal{N}\mathcal{O}_X$ is the unique quotient of Yeend's Toeplitz alg. satisfying a gauge-invariant uniqueness property:

$$\begin{array}{ccc} \mathcal{T}_{\mathrm{cov}}(X) & \xrightarrow{\psi_*} & C^*(G_\Lambda) \\ \downarrow{\scriptstyle q_{CNP}} & & \downarrow{\scriptstyle q} \\ \mathcal{N}\mathcal{O}_X & \xleftarrow[\phi]{} & C^*(\mathcal{G}_\Lambda) \end{array}$$

Key results

Lemma

Let (G, P) be quasi-lattice ordered group and X a compactly aligned product system over P of right-Hilbert A–A bimodules. Suppose either that the left action on each fibre is by injective homomorphisms, or that P is directed. Let $\psi\colon X \to B$ be an injective Nica covariant rep. of X. Fix a finite subset $F \subset P$ and fix operators $T_p \in \mathcal{K}(X_p)$ for each $p \in F$ satisfying $\sum_{p\in F} \psi^{(p)}(T_p) = 0$. Then $\sum_{p\in F} \tilde{\iota}_p^s(T_p) = 0$ for large s.

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Proposition

With the hypotheses of the lemma, let $\psi_* : \mathcal{T}_{\mathrm{cov}}(X) \to B$ be the homomorphism characterised by $\psi = \psi_* \circ i_X$. Then $\ker(\psi_*) \cap \mathcal{F} \subset \ker(q_{\mathrm{CNP}})$.

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with $B_F = \Big\{ \sum_{p \in F} i_X^{(p)}(T_p) \mid T_p \in \mathcal{K}(X_p) \text{ for each } p \in F \Big\}$. Show that B_F is a C^*-algebra for each finite $\vee$-closed subset of P.