Turbulence, representations, and trace-preserving actions

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Type of questions to consider:

Question 1: Given a separable C^*-algebra A and a separable infinite-dimensional Hilbert space H, can we classify the (non-degenerate) representations of A on H, up to unitary equivalence, using countable infinite groups as invariants?

Question 2: Given a countable group Γ, can we classify the actions of Γ on the hyperfinite II_1 factor R, up to conjugation by automorphisms of R, using countable infinite groups as invariants?

Questions on questions:

Question 1: What do we mean by "classification"?

Question 2: Why use countable infinite groups as invariants? What about countable infinite rings? Countable infinite graphs? Real numbers?

First try to define classification:

We are going to show that certain things are too complicated to be classified in any reasonable sense, so we want the definition of classification to be as weak as possible.

For the classification of representations of A on H, the simplest definition would be to have a map

$$f : \{\text{representations of } A \text{ on } H\} \to \{\text{countable infinite groups}\}$$

so that two representations are unitarily equivalent if and only if their image groups are isomorphic.

This is a bad definition:

Such maps exist by set-theoretic consideration. The cardinality of equivalence classes of representations of A on H is at most that of $\mathbb{R}$. The cardinality of isomorphism classes of countable infinite groups is that of $\mathbb{R}$. Take an injective map

$$\begin{array}{c}\{\text{equivalence classes of representations of } A \text{ on } H\} \\ \big\downarrow g \\ \{\text{isomorphisms classes of countable infinite groups}\},\end{array}$$

and for a representation π of A on H, set $f(\pi)$ to be a countable infinite group whose isomorphism class is $g([\pi])$.

Nobody would think of such a map as a classification, since practically there is no way to **compute** f.

Lesson: the map associating an invariant to an object should be computable, or reasonable.

Structure of spaces in question:

There is a natural topology on $\{$representations of A on $H\}$. Representations $\pi_j \to \pi$ if $\pi_j(a) \to \pi(a)$ in the strong operator topology for every $a \in A$. It is a *Polish space*, i.e., homeomorphic to a separable complete metric space.

To get a structure on {countable infinite groups}, we want a space to parameterize countable infinite groups.

Given a countable infinite group Γ, enumerate its elements by $g_1, g_2, \ldots$. The structure of Γ is determined by the multiplication map $\Gamma \times \Gamma \to \Gamma$. This can be written as a map $\varphi : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ for

$$g_n g_m = g_{\varphi(n,m)}.$$

Thus countable infinite groups are parameterized by the set $\{\varphi \in \mathbb{N}^{\mathbb{N}\times\mathbb{N}} : g_n g_m = g_{\varphi(n,m)} \text{ determines a group } \}$.

Another choice of the enumeration of elements of Γ would yield φ'. These two enumerations give rise to a permutation ψ of $\mathbb{N}$. In turn, ψ induces a bijection $\bar{\psi} : \mathbb{N}^{\mathbb{N}\times\mathbb{N}} \to \mathbb{N}^{\mathbb{N}\times\mathbb{N}}$. Then $\varphi' = \bar{\psi}(\varphi)$. In other words, the permutation group of $\mathbb{N}$ acts on $\mathbb{N}^{\mathbb{N}\times\mathbb{N}}$, and two elements in $\mathbb{N}^{\mathbb{N}\times\mathbb{N}}$ give rise to isomorphic groups if and only if they are in the same orbit.

The set $\mathbb{N}^{\mathbb{N}\times\mathbb{N}}$ has the natural topology of pointwise convergence, with $\mathbb{N}$ endowed with the discrete topology. $\mathbb{N}^{\mathbb{N}\times\mathbb{N}}$ is a Polish space. The set above is a G_δ subset, and hence is also a Polish space.

Second try to define classification:

It is reasonable to replace the map

$$\{\text{representations of } A \text{ on } H\} \to \{\text{countable infinite groups}\}$$

by a map

$$\{\text{representations of } A \text{ on } H\} \to \{\text{the subset of } \mathbb{N}^{\mathbb{N}\times\mathbb{N}} \text{ yielding groups}\}.$$

The meaning of this map being reasonable could be it's continuous.

A weaker requirement is that this map is *measurable*, with both spaces equipped with the σ-algebra of Borel subsets.

Why countable infinite groups?

There is no reason to use countable infinite groups specifically. One can use countable infinite rings, countable infinite graphs, etc. In each case, the objects are parameterized by a subset of something like $\mathbb{N}^{\mathbb{N}\times\mathbb{N}} \times \mathbb{N}^{\mathbb{N}\times\mathbb{N}}, \{0,1\}^{\mathbb{N}\times\mathbb{N}\times\mathbb{N}}, \ldots$.

Again, the permutation group of $\mathbb{N}$ acts on these spaces, and two elements give rise to isomorphic rings or graphs if and only if they are in the same orbit.

Definition of classification:

Let X be a measurable space. Let E be an equivalence relation on X.

For example, $X = \{$representations of A on $H\}$, and E is the unitary equivalence.

Definition (Hjorth): We say that E is *classified*, or elements of X are classified up to E-equivalence, *by countable structures*, if there is a measurable map from X to some space like $\mathbb{N}^{\mathbb{N}\times\mathbb{N}} \times \mathbb{N}^{\mathbb{N}\times\mathbb{N}}, \{0,1\}^{\mathbb{N}\times\mathbb{N}\times\mathbb{N}}, \ldots$ such that two points are E-equivalent if and only if their images are in the same orbit of the action of the permutation group of $\mathbb{N}$.

Becker and Kechris: E is classified by countable structures if and only if there are a Borel action of the permutation group of $\mathbb{N}$ on some Polish space Y and a measurable map $X \to Y$ such that two points are E-equivalent exactly when their images are in the same orbit.

An older definition:

Definition: We say that E is *classified by real numbers*, or smooth, if there is a measurable map from X to $\mathbb{R}$ such that two points are E-equivalent if and only if their images coincide.

Classification by real numbers $\Longrightarrow$ classification by countable structures.

Theorem(Glimm): The irreducible representations of A on H, up to unitary equivalence, are classified by real numbers if and only if A is of type **I**.

Theorem(Woods): The factors on H, up to isomorphism, can not be classified by real numbers.

Theorem(Ornstein): The Bernoulli shifts, up to measurable isomorphism, are classified by entropy.

Examples of classification by countable structures:

- The Halmos-von Neumann classification of discrete spectrum transformations by their sets of eigenvalues.
- The Giordano-Putnam-Skau classification of minimal homeomorphisms of the Cantor set up to strong orbit equivalence by countable ordered abelian groups.
- The Elliott classification of separable AF algebras by their K-theory.

How do we get nonclassifiability by countable structures?

Observes that the unitary group of $B(H)$ acts on $\{$representations of A on $H\}$ by conjugation. Two representations are unitarily equivalent exactly when they are in the same orbit.

Principle: if a dynamical system is too complicated, then its orbit equivalence relation can not be classified.

Turbulence:

Let G be a Polish group acting continuously on a Polish space X.

For an $x \in X$ and open sets $U \subseteq X$ and $V \subseteq G$ containing x and e, respectively, define the local orbit $\mathcal{O}(x, U, V)$ as the set of all $y \in U$ for which there are $g_1, g_2, \ldots, g_n \in V$ such that $g_k g_{k-1} \cdots g_1 x \in U$ for each $k = 1, \ldots, n-1$ and $g_n g_{n-1} \cdots g_1 x = y$.

A point $x \in X$ is *turbulent* if for all nonempty open sets $U \subseteq X$ and $V \subseteq G$ containing x and e, respectively, the closure of $\mathcal{O}(x, U, V)$ has nonempty interior.

Definition(Hjorth): The action is *turbulent* if every orbit is dense and meager and every point is turbulent. The action is *generically turbulent* if there is a G-invariant G_δ dense subset of X on which the action is turbulent.

Turbulence to nonclassifiability:

Theorem(Hjorth): If the action of G on X is generically turbulent, then the orbit equivalence relation on X can not be classified by countable structures.

Previous results of turbulence and nonclassifiability:

- Hjorth (1997): if Γ is a countably infinite group which is not a finite extension of an abelian group then the space of irreducible representations of Γ on H under the conjugation action of the unitary group $\mathcal{U}(H)$ of $B(H)$ has an invariant G_δ subset on which the action is turbulent. His argument also applies to irreducible representations of a separable non-type-I C^*-algebra A.
- Kechris and Sofronidis (2001): the conjugate actions of $\mathcal{U}(H)$ on itself and $\{T \in B(H) : T^* = T, \|T\| \leq 1\}$ are generically turbulent, where the these spaces are endowed with the strong operator topology.
- Hjorth (2001): if Γ is a countably infinite group which is not a finite extension of an abelian group, then its free weakly mixing measure-preserving actions on a standard atomless probability space (X, μ), up to conjugation by automorphisms of (X, μ), can not be classified by countable structures.

- Foreman and Weiss (2004): if Γ is a countably infinite amenable group, then the conjugate action of the automorphism group of (X, μ) on the space of free ergodic measure-preserving actions of Γ on (X, μ) is turbulent.
- Kechris (2008): for any countably infinite group Γ, its free weakly mixing measure-preserving actions on (X, μ), up to unitary conjugacy, can not be classified by countable structures.

C^*-algebra representations:

Let A be a separable C^*-algebra, H be a separable infinite-dimensional Hilbert space.

Theorem: If the isolated points of $\hat{A}$ is not dense in $\hat{A}$, then the action of $\mathcal{U}(H)$ on $\{$representations of A on $H\}$ is generically turbulent.

Theorem: If $\hat{A}$ is uncountable, then representations of A on H, up to unitary equivalence, can not be classified by countable structures.

Remark

- If $\hat{A}$ is countable, then representations of A on H, up to unitary equivalence, are classified by real numbers.
- The results of Kechris and Sofronidis correspond to the cases $A = C(\mathbb{T})$ and $A = C([0,1])$.
- The classification of all representations of A on H, and the classification of only irreducible ones, are different in this sense.

Group representations:

Take A to be the reduced group C^*-algebra.

Theorem: Let Γ be a countable infinite group. Then the action of $\mathcal{U}(H)$ on the set of unitary representations of Γ on H weakly contained in the left regular representation is generically turbulent.

Theorem: Let G be a separable noncompact Lie group. Then the unitary representations of G on H weakly contained in the left regular representation, up to unitary equivalence, can not be classified by countable structures.

Actions on the hyperfinite II_1 factor:

Let R be the hyperfinite II_1 factor.

Theorem: Let Γ be a countable infinite amenable group. Then the conjugate action of $\mathrm{Aut}(R)$ on the set of free actions of Γ on R is turbulent.

Remark:

- This is the noncommutative analogue of Foreman and Weiss' result.
- The proof uses the Connes-Narnhofer-Thirring entropy and Ocneanu's result that any two free actions of Γ on R are cocycle conjugate, with bounds on the cocycle.

Trace-preserving actions on M:

Let M be either $L^\infty(X,\tau)$ for a standard atomless probability space (X,τ) or the hyperfinite II_1 factor.

Theorem: Let Γ be a countable infinite group. Then the free weakly mixing trace-preserving actions of Γ on M, up to conjugacy by trace-preserving automorphisms of M, can not be classified by countable structures.

Remark: The case $M = L^\infty(X,\tau)$ improves Hjorth' result.

Theorem: Let G be a second countable locally compact group such that either (i) G is not amenable and the set of elements in $\hat{G}$ weakly contained in the regular representation is uncountable, or (ii) G is amenable and the set of isolated points in $\hat{G}$ is not dense. Then the weakly mixing trace-preserving actions of G on M, up to conjugacy by trace-preserving automorphisms of M, can not be classified by countable structures.

Remark: It applies to all nonamenable Lie groups, including all noncompact connected semisimple Lie groups.

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