C^*-algebras with Ideal Property and Classification

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(The talk was presented by using black board. The following is the material written on the blackboard as I can recall)

Three classes of C^*-algebras:

I. C^*-algebras of real rank zero (Brown-Pedersen)

A is called of real rank zero, if for each $x \in \mathcal{A}_{s,a}$ and $\varepsilon > 0$, $\exists\ y \in \mathcal{A}_{s,a}$ with finite spectrum such that $\|x - y\| < \varepsilon$. In other words, there are finite many mutually orthogonal projections $p_1, p_2, \cdots, p_n$ and real numbers $\lambda_1, \lambda_2, \cdots, \lambda_n$ such that

$$\|x - \sum_{i=1}^{n} \lambda_i p_i\| < \varepsilon$$

II. Simple C^*-algebras: C^*-algebra $\mathcal{A}$ without any nontrivial closed two sided ideal.

III. C^*-algebra with ideal property (Elliott), Any nontrivial closed two sided ideal $I \subset \mathcal{A}$ is generated by the projections inside the ideal.

Obviously Class II $\subset$ Class III

Also Class I $\subset$ Class III

(proof: Any ideal of real rank zero C^*-algebra is also of real rank zero. So in the approximation $\|x - \sum_{i=1}^{n} \lambda_i p_i\| < \varepsilon$, if $x \in I \subset \mathcal{A}$, then p_i can be chosen to be in I.)

For purely infinite algebra, Class II = Class III (Classification : R$\ddot{o}$rdam, Elliott-R$\ddot{o}$rdam, Kerchberg, Phillips, ...)

Focus on stably finite C^*-algebras.

Last twenty years: There are many significant classification results:

For Class I: Elliott, Su, E-G, Lin, E-G-Lin-Pasnicu, Dadarlat-G, Eilers, D-Loring, D-G.

For Class II: Elliott, Li, Jiang-Su, G, E-G-Li.

Pasnicu studied Class III intensively. But only recently classification becomes possible.

An <u>AH-algebra</u> $\mathcal{A}$ is an inductive limit

$$\mathcal{A}_1 \xrightarrow{\phi_1} \mathcal{A}_2 \xrightarrow{\phi_2} \cdots \longrightarrow \mathcal{A}_n \xrightarrow{\phi_n} \cdots \longrightarrow \mathcal{A},$$

where $\mathcal{A}_n = \bigoplus_{i=1}^{k_n} M_{[n,i]}(C(X_{n,i}))$, $X_{n,i}$ are compact metric spaces, and $[n,i]$ are positive integers.

Notation: $\phi_{n,m}: \mathcal{A}_n \to \mathcal{A}_m$,

$\phi_{n,m} = \phi_{m-1} \circ \phi_{m-2} \circ \cdots \phi_n : \mathcal{A}_n \to \mathcal{A}_m$.

Elliott-G-Li: One can replace $X_{n,i}$ by connected simplicial complex and ϕ_n by injective homomorphisms.

No dimension growth: $\sup \dim(X_{n,i}) < \infty$.

In general, a homomorphism $\phi : M_n(C(X)) \to M_m(C(Y))$ can be written as

$$\phi(f)(y) = u_y \begin{pmatrix} f(x_1(y)) & & & & & & \\ & f(x_2(y)) & & & & & \\ & & \ddots & & & & \\ & & & f(x_\bullet(y)) & & & \\ & & & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 0 \end{pmatrix} u_y^*$$

Denote: $Sp\phi|_y = \{x_1(y), x_2(y), \cdots, x_\bullet(y)\} \subset X$ counting multiplicity.

For X metric space, $F \subset X$ a closed subset, Denote $B\varepsilon(F) = \{y \in X,\ dist(y, F) < \varepsilon\}$.

Theorem. Assume $\phi_{n,m}$ injective and no dimension growth.

1 (Su, Elliott-G) $\mathcal{A}$ is of real rank zero $\Longleftrightarrow$ $\forall \mathcal{A}_n,\ \varepsilon > 0,\ \exists m$ such that for any $y, y' \in X_{m,j},\ Sp\phi^{i,j}_{n,m}|_y$ and $Sp\phi^{i,j}_{n,m}|_{y'}$ can be paired within ε.

2. (Dadarlat-Nagy-Nemethi-Pasnicu) $\mathcal{A}$ is simple $\Longleftrightarrow$ $\forall \mathcal{A}_n,\ \varepsilon > 0,\ \exists m$ such that for any $y \in X_{m,j},\ Sp\phi^{i,j}_{n,m}|_y$ is ε-dense in $X_{n,i}$. That is, $X_{n,i} \subseteq B_\varepsilon(\phi^{i,j}_{n,m}|_y)\ \forall y \in X_{m,j}$.

3. (Pasnicu) $\mathcal{A}$ has ideal property $\Longleftrightarrow$ $\forall \mathcal{A}_n,\ \varepsilon > 0,\ \exists m$ such that for any $y, y' \in X_{m,j}$,

$$Sp\phi^{i,j}_{n,m}|_{y'} \subseteq B_\varepsilon(\phi^{i,j}_{n,m}|_y).$$

Classification of real rank zero AH-algebras with no dimension growth (Elliott-G, Dadarlat-G)

Classification of simple AH-algebras with no dimension growth (G, Elliott-G-Li)

How about AH-algebras with ideal property? Why do we need classification for this class?

Three reasons:

1. It unities real rank zero C^*-algebras and simple C^*-algebras.

2. There are some natural C^*-algebras with ideal property which do not belong to the class of real rank zero C^*-algebras and the class of simple C^*-algebras.

3. There are some C^*-algebras, for which it is easy to prove they have ideal property. But it is difficult to prove that they have real rank zero. On the other hand, if the classification theorem holds for C^*-algebra with ideal property, then it will imply the C^*-algebras have real rank zero.

Proposition [Sierakowski]. Let $(\mathcal{A}, G, \alpha)$ be a $C*$-dynamical system, where G is a discrete amenable group and the action of G on $\hat{\mathcal{A}}$ is free, then $\mathcal{A} \rtimes_\alpha G$ has ideal property provided that $\mathcal{A}$ has ideal property. In general, such $\mathcal{A}$ is not of real rank zero and also not simple.

Ex 1. (Pasnicu) $\mathcal{A} = C(X)$, $\dim(X) = 0$, $G = \mathbb{Z}$ acts on X freely, then $C(X) \rtimes_\alpha \mathbb{Z}$ has ideal property. If the action is not minimal, then $C(X) \rtimes_\alpha \mathbb{Z}$ is not simple. Also it is not known whether $C(X) \rtimes_\alpha \mathbb{Z}$ is of real rank zero. But the classification for this class of C^*-algebras, implies that $C(X) \rtimes_\alpha \mathbb{Z}$ is of real rank zero.

Ex 2. Let $\alpha : S^3 \to S^3$ a minimum diffeomorphism, $\beta : X \to X$ a free action which is not minimum, where X is a locally compact metric space with $\dim(X) = 0$. Then $C_0(S^3 \times X) \rtimes_{\alpha\times\beta} (\mathbb{Z} \oplus \mathbb{Z})$ has ideal property. But in general, is not of real rank zero, also not simple.

Invariant: Involve order on K-theory and order on $mod-p$–K-theory. Trace spaces (of $\mathcal{A}$ and of ideals of $\mathcal{A}$), and maps between trace spaces. Call it $Inv(\mathcal{A})$

(Stevens 1994) Classification for the case $X_{n,i} = [0,1]$, with extra restrictions of $\mathcal{A}$ unital and $\mathcal{A}$ is approximately divisible.

(The proof is quite special, can not be generalized).

Ji-Jiang (2007): Remove the restrictions of unital and approximately divisible.

This paper is a significant paper, introduced several techniques, which can be applied to general case.

Jiang-Wang (2009): Classifies inductive limit of splitting interval algebras with ideal property: $\mathcal{A}_n^i$ is something like

$$\left\{ f \in M_{[n,i]}(C[0,1]),\ f(0) = \begin{pmatrix} \square & & \\ & \square & \\ & & \square \end{pmatrix},\ f(1) = \begin{pmatrix} \square & & & \\ & \square & & \\ & & \square & \\ & & & \square \end{pmatrix} \right\}$$

Theorem Gong-Jiang-Li-Pasnicu (2007) An AH-algebra with ideal property with no dimension growth can be rewritten as AH-algebras with spaces $X_{n,i}$ being $S^1, T_{II,k}, T_{III,k}$ and S^2, where $T_{II,k}$ is a connected 2-dimensional simplicial complex with $H^2(T_{II,k}) = \mathbb{Z}/k\mathbb{Z}, H^1(T_{III,k}) = 0$ and $T_{III,k}$ is a connected 3-dimensional simplicial complex with $H^3(T_{III,k}) = \mathbb{Z}/k\mathbb{Z}, H^1(T_{III,k}) = 0 = H^2(T_{III,k})$.

The case of real rank zero AH-algebras is due to Dadarlat-G.

The case of simple AH-algebras is due to G.