Cubic Column Relations in Truncated Moment Problems

(joint work with Seonguk Yoo)

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The Truncated Complex Moment Problem

- Given $\gamma : \gamma_{00}, \gamma_{01}, \gamma_{10}, \ldots, \gamma_{0,2n}, \ldots, \gamma_{2n,0}$, with $\gamma_{00} > 0$ and $\gamma_{ji} = \bar{\gamma}_{ij}$, the **TCMP** entails finding a positive Borel measure μ supported in the complex plane $\mathbb{C}$ such that

$$\gamma_{ij} = \int \bar{z}^i z^j \, d\mu \qquad (0 \leq i + j \leq 2n);$$

  μ is called a **rep. meas.** for γ.

- In earlier joint work with L. Fialkow,
- We have introduced an approach based on matrix positivity and extension, combined with a new "functional calculus" for the columns of the associated **moment matrix**.

- We have shown that when the TCMP is of **flat data type**, a solution always exists; this is compatible with our previous results for

$$\begin{array}{ll} \text{supp } \mu \subseteq \mathbb{R} & \text{(Hamburger TMP)} \\ \text{supp } \mu \subseteq [0,\infty) & \text{(Stieltjes TMP)} \\ \text{supp } \mu \subseteq [a,b] & \text{(Hausdorff TMP)} \\ \text{supp } \mu \subseteq \mathbb{T} & \text{(Toeplitz TMP)} \end{array}$$

- Along the way we have developed new machinery for analyzing TMP's in **one or several real or complex variables**. For simplicity, in this talk we focus on **one complex variable or two real variables**, although several results have multivariable versions.

- Our techniques also give concrete algorithms to provide finitely-atomic rep. meas. whose atoms and densities can be explicitly computed.
- We have fully resolved, among others, the cases

$$\bar{Z} = \alpha 1 + \beta Z$$

and

$$Z^k = p_{k-1}(Z, \bar{Z}) \ \ (1 \leq k \leq [\frac{n}{2}] + 1; \deg p_{k-1} \leq k-1).$$

- We obtain applications to quadrature problems in numerical analysis.
- We have obtained a duality proof of a generalized form of the Tchakaloff-Putinar Theorem on the existence of quadrature rules for positive Borel measures on $\mathbb{R}^d$.

Positivity of Block Matrices

Theorem

(Smul'jan, 1959)

$$\begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \geq 0 \Leftrightarrow \begin{cases} A \geq 0 \\ B = AW \\ C \geq W^*AW \end{cases}.$$

Moreover, $\operatorname{rank} \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} = \operatorname{rank} A \Leftrightarrow C = W^*AW.$

Corollary

Assume $\operatorname{rank}\begin{pmatrix} A & B \\ B^* & C \end{pmatrix} = \operatorname{rank} A$. *Then*

$$A \geq 0 \Leftrightarrow \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \geq 0.$$

Basic Positivity Condition

$\mathcal{P}_n$: polynomials p in z and $\bar{z}$, $\deg p \leq n$

Given $p \in \mathcal{P}_n$, $p(z, \bar{z}) \equiv \sum_{0 \leq i+j \leq n} a_{ij} \bar{z}^i z^j$,

$$0 \leq \int \mid p(z, \bar{z}) \mid^2 d\mu(z, \bar{z})$$

$$= \sum_{ijk\ell} a_{ij} \bar{a}_{k\ell} \int \bar{z}^{i+\ell} z^{j+k} d\mu(z, \bar{z})$$

$$= \sum_{ijk\ell} a_{ij} \bar{a}_{k\ell} \gamma_{i+\ell, j+k}.$$

- To understand this **"matricial" positivity**, we introduce the following lexicographic order on the rows and columns of $M(n)$:

$$1, Z, \bar{Z}, Z^2, \bar{Z}Z, \bar{Z}^2, \ldots$$

Define $M[i,j]$ as in

$$M[3,2] := \begin{pmatrix} \gamma_{32} & \gamma_{41} & \gamma_{50} \\ \gamma_{23} & \gamma_{32} & \gamma_{41} \\ \gamma_{14} & \gamma_{23} & \gamma_{32} \\ \gamma_{05} & \gamma_{14} & \gamma_{23} \end{pmatrix}$$

Then

$$(\textbf{"matricial" positivity}) \quad \sum_{ijk\ell} a_{ij}\bar{a}_{k\ell}\gamma_{i+\ell,j+k} \geq 0$$

$$\Leftrightarrow M(n) \equiv M(n)(\gamma) := \begin{pmatrix} M[0,0] & M[0,1] & ... & M[0,n] \\ M[1,0] & M[1,1] & ... & M[1,n] \\ ... & ... & ... & \dots \\ M[n,0] & M[n,1] & \dots & M[n,n] \end{pmatrix} \geq 0.$$

For example,

- $$M(1) = \begin{pmatrix} \gamma_{00} & \gamma_{01} & \gamma_{10} \\ \gamma_{10} & \gamma_{11} & \gamma_{20} \\ \gamma_{01} & \gamma_{02} & \gamma_{11} \end{pmatrix},$$

- $$M(2) = \begin{pmatrix} \gamma_{00} & \gamma_{01} & \gamma_{10} & \gamma_{02} & \gamma_{11} & \gamma_{20} \\ \gamma_{10} & \gamma_{11} & \gamma_{20} & \gamma_{12} & \gamma_{21} & \gamma_{30} \\ \gamma_{01} & \gamma_{02} & \gamma_{11} & \gamma_{03} & \gamma_{12} & \gamma_{21} \\ \gamma_{20} & \gamma_{21} & \gamma_{12} & \gamma_{22} & \gamma_{31} & \gamma_{40} \\ \gamma_{11} & \gamma_{12} & \gamma_{21} & \gamma_{13} & \gamma_{22} & \gamma_{31} \\ \gamma_{02} & \gamma_{03} & \gamma_{12} & \gamma_{04} & \gamma_{13} & \gamma_{22} \end{pmatrix}.$$

In general,

$$M(n+1) = \begin{pmatrix} M(n) & B \\ B^* & C \end{pmatrix}$$

Similarly, one can build $M(\infty)$.

In the real case, $\mathcal{M}(n)_{ij} := \gamma_{i+j}, \ i,j \in \mathbb{Z}_+^2$.

Positivity Condition is not sufficient:

By modifying an example of K. Schmüdgen, we have built a family $\gamma_{00}, \gamma_{01}, \gamma_{10}, ..., \gamma_{06}, ..., \gamma_{60}$ with positive invertible moment matrix $M(3)$ but **no** rep. meas. But this can also be done for $n = 2$.

Functional Calculus

For $p \in \mathcal{P}_n,\ p(z,\bar{z}) \equiv \sum_{0 \le i+j \le n} a_{ij} \bar{z}^i z^j$ define

$$p(Z,\bar{Z}) := \sum a_{ij} \bar{Z}^i Z^j.$$

If there exists a rep. meas. μ, then

$$p(Z,\bar{Z}) = 0 \Leftrightarrow \text{supp } \mu \subseteq \mathcal{Z}(p).$$

The following is our analogue of recursiveness for the TCMP

(RG) **If** $p, q, pq \in \mathcal{P}_n$, **and** $p(Z,\bar{Z}) = 0$,

then $(pq)(Z,\bar{Z}) = 0$.

Singular TMP; Real Case

- Given a finite family of moments, build moment matrix
- Identify all column relations
- Build algebraic variety $\mathcal{V}$
- Consider the ideal $\mathcal{I} \subseteq \mathcal{P} \equiv \mathbb{R}[x, y]$ generated by poly's arising from column relations
- The ideal $\mathcal{I}$ is always radical, i.e., $\mathcal{I} = \sqrt{\mathcal{I}} := \{f \in \mathcal{P} : f^k \in \mathcal{I} \text{ for some } k \geq 1\}$
- If $\mathcal{V}$ is finite, then $\mathcal{I}$ is zero-dimensional, i.e., $V(\mathcal{I})$ is finite, where $V(\mathcal{I}) := \{x \in \mathbb{C}^2 : f(x) = 0 \text{ for all } f \in \mathcal{I}\}$

Singular TMP

- Always true:

$$r := \text{rank}\,\mathcal{M}(n) \leq \text{card supp } \mu \leq v := \text{card}\,\mathcal{V}(\beta),$$

  so if the variety is finite there's a natural candidate for supp μ

- Finite rank case
- Flat case
- Extremal case
- Recursively generated relations
- Build positive extension, repeat, and eventually flatten
- General case.

First Existence Criterion

Theorem

(RC-L. Fialkow, 1998) Let γ be a truncated moment sequence. TFAE:

(i) γ has a rep. meas.;

(ii) γ has a rep. meas. with moments of all orders;

(iii) γ has a compactly supported rep. meas.;

(iv) γ has a finitely atomic rep. meas. (with at most $(n+2)(2n+3)$ atoms);

(v) $M(n) \geq 0$ and for some $k \geq 0$ $M(n)$ admits a positive extension $M(n+k)$, which in turn admits a flat (i.e., rank-preserving) extension $M(n+k+1)$ (here $k \leq 2n^2+6n+6$).

Case of Flat Data

Recall: If μ is a rep. meas. for $M(n)$, then rank $M(n) \leq$ card supp μ.

γ is flat if $M(n) = \begin{pmatrix} M(n-1) & M(n-1)W \\ W^* M(n-1) & W^* M(n-1)W \end{pmatrix}$.

Theorem

(RC-L. Fialkow, 1996) If γ is flat and $M(n) \geq 0$, then $M(n)$ admits a unique flat extension of the form $M(n+1)$.

Theorem

(RC-L. Fialkow, 1996) The truncated moment sequence γ has a rank $M(n)$*-atomic rep. meas. if and only if $M(n) \geq 0$ and $M(n)$ admits a flat extension $M(n+1)$.*

To find μ concretely, let $r :=$rank $M(n)$ and look for the relation

$$Z^r = c_0 1 + c_1 Z + ... + c_{r-1} Z^{r-1}.$$

We then define

$$p(z) := z^r - (c_0 + ... + c_{r-1} z^{r-1})$$

and solve the Vandermonde equation

$$\begin{pmatrix} 1 & \cdots & 1 \\ z_0 & \cdots & z_{r-1} \\ \cdots & \cdots & \cdots \\ z_0^{r-1} & \cdots & z_{r-1}^{r-1} \end{pmatrix} \begin{pmatrix} \rho_0 \\ \rho_1 \\ \cdots \\ \rho_{r-1} \end{pmatrix} = \begin{pmatrix} \gamma_{00} \\ \gamma_{01} \\ \cdots \\ \gamma_{0r-1} \end{pmatrix}.$$

Then

$$\mu = \sum_{j=0}^{r-1} \rho_j \delta_{z_j}.$$

The Quartic Moment Problem

Recall the lexicographic order on the rows and columns of $M(2)$:

$$1, Z, \bar{Z}, Z^2, \bar{Z}Z, \bar{Z}^2$$

- $Z = A\,1$ (Dirac measure)
- $\bar{Z} = A\,1 + B\,Z$ (supp $\mu \subseteq$ line)
- $Z^2 = A\,1 + B\,Z + C\,\bar{Z}$ (flat extensions always exist)
- $\bar{Z}Z = A\,1 + B\,Z + C\,\bar{Z} + D\,Z^2$

$$\begin{aligned} D &= 0 \Rightarrow \bar{Z}Z = A\,1 + B\,Z + \bar{B}\,\bar{Z} \text{ and } C = \bar{B} \\ &\Rightarrow (\bar{Z} - B)(Z - \bar{B}) = A + |B|^2 \\ &\Rightarrow \bar{W}W = 1 \text{ (circle), for } W := \frac{Z - \bar{B}}{\sqrt{A + |B|^2}}. \end{aligned}$$

The functional calculus we have constructed is such that $p(Z, \bar{Z}) = 0$ implies supp $\mu \subseteq \mathcal{Z}(p)$.

When $\{1, Z, \bar{Z}, Z^2, \bar{Z}Z\}$ is a basis for $\mathcal{C}_{M(2)}$, the associated algebraic variety is the zero set of a real quadratic equation in $x := \mathrm{Re}[z]$ and $y := \mathrm{Im}[z]$.

Using the flat data result, one can reduce the study to cases corresponding to the following four real conics:

(a)	$\bar{W}^2 = -2iW + 2i\bar{W} - W^2 - 2\bar{W}W$	parabola; $y = x^2$
(b)	$\bar{W}^2 = -4i1 + W^2$	hyperbola; $yx = 1$
(c)	$\bar{W}^2 = W^2$	pair of intersect. lines; $yx = 0$
(d)	$\bar{W}W = 1$	unit circle; $x^2 + y^2 = 1$.

Theorem QUARTIC

(RC-L. Fialkow, 2005) Let $\gamma^{(4)}$ be given, and assume $M(2) \geq 0$ and $\{1, Z, \bar{Z}, Z^2, \bar{Z}Z\}$ is a basis for $\mathcal{C}_{M(2)}$. Then $\gamma^{(4)}$ admits a rep. meas. μ. Moreover, it is possible to find μ with card supp $\mu =$ rank $M(2)$, *except in some cases when $\mathcal{V}(\gamma^{(4)})$ is a pair of intersecting lines, in which cases there exist μ with* card supp $\mu \leq 6$.

Corollary

Assume that $M(2) \geq 0$ and that rank $M(2) \leq$ card $\mathcal{V}(\gamma^{(4)})$. *Then $M(2)$ admits a representing measure.*

Extremal MP; $r = v$

The *algebraic variety* of β is

$$\mathcal{V} \equiv \mathcal{V}_\beta := \bigcap_{p \in \mathcal{P}_n, \hat{p} \in \ker \mathcal{M}(n)} \mathcal{Z}_p,$$

where $\mathcal{Z}_p = \{x \in \mathbb{R}^d : p(x) = 0\}$.

- If β admits a representing measure μ, then

$$p \in \mathcal{P}_n \text{ satisfies } \hat{p} \in \ker \mathcal{M}(n) \Leftrightarrow \text{supp } \mu \subseteq \mathcal{Z}_p$$

Thus supp $\mu \subseteq \mathcal{V}$, so $r :=$ rank $\mathcal{M}(n)$ and $v :=$ card $\mathcal{V}$ satisfy

$$r \leq \text{card supp } \mu \leq v.$$

If $p \in \mathcal{P}_{2n}$ and $p|_{\mathcal{V}} \equiv 0$, then $\Lambda(p) = \int p \, d\mu = 0$.

Here Λ is the Riesz functional, given by $\Lambda(\bar{z}^i z^j) := \gamma_{ij}$

Basic necessary conditions for the existence of a representing measure

$$\text{(Positivity) } \mathcal{M}(n) \geq 0 \tag{8.1}$$

$$\text{(Consistency) } p \in \mathcal{P}_{2n},\ p|_{\mathcal{V}} \equiv 0 \Longrightarrow \Lambda(p) = 0 \tag{8.2}$$

$$\text{(Variety Condition) } r \leq v, \text{ i.e., } \operatorname{rank} \mathcal{M}(n) \leq \operatorname{card} \mathcal{V}. \tag{8.3}$$

Consistency implies

$$\text{(Recursiveness) } p, q, pq \in \mathcal{P}_n,\ \hat{p} \in \ker \mathcal{M}(n) \Longrightarrow \widehat{pq} \in \ker \mathcal{M}(n). \tag{8.4}$$

Previous results:

- For $d = 1$ (the T *Hamburger* MP for $\mathbb{R}$), positivity and recursiveness are sufficient
- For $d = 2$, there exists $\mathcal{M}(3) > 0$ for which β has no representing measure
- In general, *Positivity*, *Consistency* and the *Variety Condition* are **not** sufficient.

QUESTION C

Suppose $\mathcal{M}(n)(\beta)$ is singular. If $\mathcal{M}(n)$ is positive, β is consistent, and $r \leq v$, does β admit a representing measure?

The next result gives an affirmative answer to Question C in the *extremal* case, i.e., $r = v$.

Theorem EXT

(RC, L. Fialkow and M. Möller, 2005) For $\beta \equiv \beta^{(2n)}$ **extremal**, *i.e.,* $r = v$, *the following are equivalent:*

(i) β *has a representing measure;*

(ii) β *has a unique representing measure, which is* rank $\mathcal{M}(n)$*-atomic (minimal);*

(iii) $\mathcal{M}(n) \geq 0$ *and* β *is consistent.*

Cubic Column Relations

Since we know how to solve the singular Quartic MP, WLOG we will assume $M(2) > 0$.

Recall

Theorem A

(RC-L. Fialkow) If $M(n)$ admits a column relation of the form $Z^k = p_{k-1}(Z, \bar{Z})$ $(1 \leq k \leq \left[\frac{n}{2}\right] + 1$ and $\deg\ p_{k-1} \leq k-1)$, then $M(n)$ admits a flat extension $M(n+1)$, and therefore a representing measure.

Now, if $k = 3$, Theorem A can be used only if $n \geq 4$. Thus, one strategy is to somehow extend $M(3)$ to $M(4)$ and preserve the column relation $Z^3 = p_2(Z, \bar{Z})$. This requires checking that the C block in the extension satisfies the Toeplitz condition, something highly nontrivial.

Here's a different approach:

We'd like to study the case of harmonic poly's: $q(z,\bar{z}) := f(z) - \overline{g(z)}$, with deg $q = 3$.

Recall that rank $M(n) \leq$ card $\mathcal{Z}(q)$

so of special interest is the case when card $\mathcal{Z}(q) \geq 7$, since otherwise the TMP admits a flat extension, or has no representing measure. In the case when $g(z) \equiv z$, we have

Lemma

(Wilmshurst '98, Sarason-Crofoot, '99, Khavinson-Swiatek, '03)

$$\text{card } \mathcal{Z}(f(z) - \bar{z}) \leq 7.$$

- To get 7 points is not easy, as most complex cubic harmonic poly's tend to have 5 or fewer zeros. One way to maximize the number of zeros is to impose symmetry conditions on the zero set K. Also, the substitution $w = z + b/3$ (which produces an equivalent TMP) transforms a cubic $z^3 + bz^2 + cz + d$ into $w^3 + \tilde{c}w + \tilde{d}$; WLOG, we always assume that there's no quadratic term in the analytic piece.
- Now, for a poly of the form $z^3 + \alpha z + \beta \bar{z}$, it is clear that $0 \in K$ and that $z \in K \Rightarrow -z \in K$. Another natural condition is to require that K be symmetric with respect to the line $y = x$, which in complex notation is $z = i\bar{z}$. When this is required, we obtain $\alpha \in i\mathbb{R}$ and $\beta \in \mathbb{R}$. Thus, the column relation becomes $Z^3 = itZ + u\bar{Z}$, with $t, u \in \mathbb{R}$.

  Under these conditions, one needs to find only two points, one on the line $y = x$, the other outside that line.

We thus consider the harmonic polynomial $q_7(z,\bar{z}) := z^3 - itz - u\bar{z}$.

PROPOSITION

(RC-S. Yoo, '09) card $\mathcal{Z}(q_7) = 7$. *In fact, for* $0 < |u| < t < 2|u|$,

$$\mathcal{Z}(q_7) = \{0, p+iq, q+ip, -p-iq, -q-ip, r+ir, -r-ir\},$$

where $p,q,r > 0$, $p^2+q^2 = u$ *and* $r^2 = \frac{t-u}{2}$.

To prove this result, we first identify the two real poly's Re $q_7 = x^3 - 3xy^2 + ty - ux$ and Im $q_7 = -y^3 + 3x^2y - tx + uy$ and calculate $Resultant(Req_7, Imq_7, y)$, which is the determinant of the Sylvester matrix, i.e.,

$$\det \begin{pmatrix} -3x & t & x^3 - ux & 0 & 0 \\ 0 & -3x & t & x^3 - ux & 0 \\ 0 & 0 & -3x & t & x^3 - ux \\ -1 & 0 & 3x^2 + u & -tx & 0 \\ 0 & -1 & 0 & 3x^2 + u & -tx \end{pmatrix}$$

$$= \quad x\left(u - t + 2x^2\right)\left(u + t + 2x^2\right)\left(16x^4 - 16x^2u + t^2\right).$$

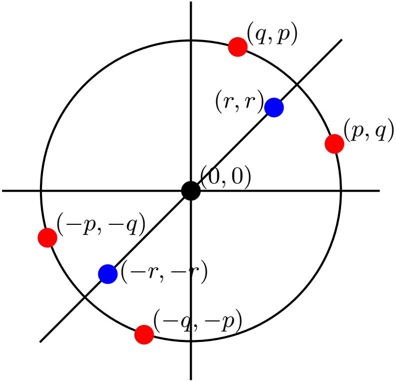

FIGURE 1. The 7-point set $\mathcal{Z}(q_7)$, where $r = \sqrt{\frac{t-u}{2}}$, $p = \frac{1}{2}(2u + \sqrt{4u^2 - t^2})$ and $p^2 + q^2 = u$

The fact that q_7 has the maximum number of zeros predicted by the Lemma is significant to us, in that each sextic TMP with invertible $M(2)$ and a column relation of the form $q_7(Z, \bar{Z}) = 0$ either does not admit a representing measure or is necessarily extremal.

As a consequence, the existence of a representing measure will be established once we prove that such a TMP is consistent. This means that for each poly p of degree at most 6 that vanishes on $\mathcal{Z}(q_7)$ we must verify that $\Lambda(p) = 0$.

Since rank $M(3) = 7$, there must be another column relation besides $q_7(Z, \bar{Z}) = 0$. Clearly the columns

$$1, Z, \bar{Z}, Z^2, \bar{Z}Z, \bar{Z}^2, \bar{Z}Z^2$$

must be linearly independent (otherwise $M(3)$ would be a flat extension of $M(2)$), so the new column relation must involve $\bar{Z}Z^2$ and $\bar{Z}^2Z$. An analysis using the properties of the functional calculus shows that, in the presence of a representing measure, the new column relation must be

$$\bar{Z}^2Z + i\bar{Z}Z^2 - iuZ - u\bar{Z} = 0.$$

Notation

In what follows, $\mathbb{C}_6[z,\bar{z}]$ will denote the space of complex polynomials in z and $\bar{z}$ of degree at most 6, and let

$$\begin{aligned} q_{LC}(z,\bar{z}) &:= \bar{z}^2 z + i\bar{z}z^2 - iuz - u\bar{z} \\ &= i(z - i\bar{z})(\bar{z}z - u). \end{aligned}$$

Observe that the zero set of q_{LC} is the union of a line and a circle, and that $\mathcal{Z}(q_7) \subset \mathcal{Z}(q_{LC})$.

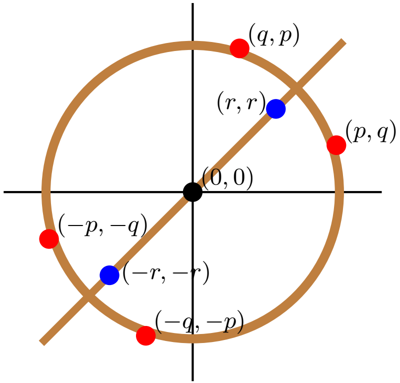

FIGURE 2. The sets $\mathcal{Z}(q_7)$ and $\mathcal{Z}(q_{LC})$

Main Theorem

Let $M(3) \geq 0$, *with* $M(2) > 0$ *and* $q_7(Z, \bar{Z}) = 0$. *There exists a representing measure for* $M(3)$ *if and only if*

$$\begin{cases} \Lambda(q_{LC}) & = & 0 \\ \Lambda(zq_{LC}) & = & 0. \end{cases} \tag{8.5}$$

Equivalently,

$$\begin{cases} Re\ \gamma_{12} - Im\ \gamma_{12} = u(Re\ \gamma_{01} - Im\ \gamma_{01}) & = & 0 \\ \gamma_{22} = (t+u)\gamma_{11} - 2u\ Im\ \gamma_{02} & = & 0. \end{cases}$$

Equivalently,

$$q_{LC}(Z, \bar{Z}) = 0 \tag{8.6}$$

Proof. ($\Longrightarrow$) Let μ be a representing measure. We know that $7 \leq \text{rank } M(3) \leq \text{card supp } \mu \leq \text{card } \mathcal{Z}(q_7) = 7$, so that $\text{supp } \mu = \mathcal{Z}(q_7)$ and $\text{rank } M(3) = 7$. Thus,

$$\Lambda(q_7) = \int q_7 \, d\mu = 0.$$

Similarly, since $\text{supp } \mu \subseteq \mathcal{Z}(q_{LC})$, we also have

$$\Lambda(q_{LC}) = \Lambda(zq_{LC}) = 0,$$

as desired.

($\Longleftarrow$) On $\mathcal{Z}(q_7)$ we have $z^3 = itz + u\bar{z}$. Using this relation and (8.5), we can prove that $\Lambda(\bar{z}^i z^j q_{LC}) = 0$ for all $0 \leq i + j \leq 3$. For example,

$$\begin{aligned} \bar{z}q_{LC} - izq_{LC} &= (\bar{z} - iz)(\bar{z}^2 z + i\bar{z}z^2 - iuz - u\bar{z}) \\ &= -uz^2 + \bar{z}z^3 - u\bar{z}^2 + \bar{z}^3 z \\ &= -uz^2 + \bar{z}(itz + u\bar{z}) - u\bar{z}^2 + (-it\bar{z} + uz)z \\ &= 0, \end{aligned}$$

and therefore $\Lambda(\bar{z}q_{LC}) = i\Lambda(zq_{LC}) = 0$. It follows that for $f, g, h \in \mathbb{C}_3[z, \bar{z}]$ we have $\Lambda(fq_7 + g\bar{q}_7 + hq_{LC}) = 0$. Consistency will be established once we show that all degree-six polynomials vanishing in $\mathcal{Z}(q_7)$ are of the form $fq_7 + g\bar{q}_7 + hq_{LC}$.

Proposition (Representation of Polynomials)

Let $\mathcal{P}_6 := \{p \in \mathbb{C}_6[z,\bar{z}] : p|_{\mathcal{Z}(q_7)} \equiv 0\}$ *and let*
$\mathcal{I} := \{p \in \mathbb{C}_6[z,\bar{z}] : p = fq_7 + g\bar{q}_7 + hq_{LC}$ *for some* $f, g, h \in \mathbb{C}_3[z,\bar{z}]\}$.
Then $\mathcal{P}_6 = \mathcal{I}$.

Proof. Clearly, $\mathcal{I} \subseteq \mathcal{P}_6$. We shall show that $\dim \mathcal{I} = \dim \mathcal{P}_6$. Let $T : \mathbb{C}^{30} \longrightarrow \mathbb{C}_6[z,\bar{z}]$ be given by

$$(a_{00}, \cdots, a_{30}, b_{00}, \cdots, b_{30}, c_{00}, \cdots, c_{30}) \longmapsto$$

$$\begin{aligned} &(a_{00} + a_{01}z + a_{10}\bar{z} + \cdots + a_{30}\bar{z}^3)q_7 \\ +&(b_{00} + b_{01}z + b_{10}\bar{z} + \cdots + b_{30}\bar{z}^3)\bar{q}_7 \\ +&(c_{00} + c_{01}z + c_{10}\bar{z} + \cdots + c_{30}\bar{z}^3)q_{LC}. \end{aligned}$$

Observe that $\mathcal{I} = \text{Ran } T$, so that $\dim \mathcal{I} = \text{rank } T$. To determine rank T, we first determine $\dim \ker T$. Using Gaussian elimination, we prove that $\dim \ker T = 9$ whenever $ut \neq 0$. It follows that rank $T = 21$, that is, $\dim \mathcal{I} = 21$.

Now consider the evaluation map $S : \mathbb{C}_6[z, \bar{z}] \longrightarrow \mathbb{C}^7$ given by

$$\begin{aligned} S(p(z,\bar{z})) \quad := \quad & (p(w_0,\bar{w}_0), p(w_1,\bar{w}_1), p(w_2,\bar{w}_2), \\ & p(w_3,\bar{w}_3), p(w_4,\bar{w}_4), p(w_5,\bar{w}_5), p(w_6,\bar{w}_6)). \end{aligned}$$

Using Lagrange Interpolation, it is esay to verify that S is onto. Moreover, $\ker\ S = \mathcal{P}_6$. Since $\dim\ \mathbb{C}_6[z,\bar{z}] = 28$, it follows that $\dim\ \ker\ S = 21$, and a fortiori that $\dim\ \mathcal{P}_6 = 21$. We have now established that $\dim\ \mathcal{I} = \dim\ \mathcal{P}_6$, that is, $\mathcal{I} = \mathcal{P}_6$.

Summary

- Given a finite family of moments, build moment matrix
- Identify all column relations, and build algebraic variety $\mathcal{V}$
- Consider the ideal generated by poly's arising from column relations
- Always true: $r \leq \text{card supp } \mu \leq v$
- Finite rank case; flat case
- Quartic Case
- Extremal case
- Harmonic cubic poly's in Sextic Case
- General singular case
- Invertible case still a big mystery...