Extending a normal state from a MASA of a von Neumann algebra

by Charles Akemann

David Sherman and I are just finishing a closely related paper titled *Conditional Expectations onto Maximal Abelian Subalgebras* from which most of the ideas for this talk were taken, but I should be blamed for any errors, NOT David.

Abstract: Ever since the 1959 Kadison-Singer paper, there has been substantial interest in extending **pure** states from maximal abelian C*-subalgebras (MASAs) of C*-algebras. In this talk I will change the problem and ask the following.

Question: Given a MASA A of a von Neumann algebra N and a **normal** state f_0 of A, what can be said about the set S of all state extensions of f_0 to N?

There are some large gaps in my knowledge about this question. I will talk about partial results and useful techniques, and I will show how this question relates to various other things like conditional expectations. I promise to leave you with many interesting unsolved problems.

We know that S is a weak* compact convex subset of N^* and here are some sample questions where the answers are much less clear.

1. When is S a singleton?
2. When is S finite dimensional?
3. When is S norm compact?
4. When does S contain a non-normal element?
5. When does S contain a singular element?
6. What are the weak*-closed faces of S, especially he extreme points?
7. Let S_A denote the set of elements $g \in S$ such that A is contained in the centralizer of g, i.e. for all $a \in A$ and $b \in N, g(ba) = g(ab)$. What are the answers to questions 1-6 for S_A.
8. For what pairs N, A can we say that a state f of N is normal iff f_0 is normal?

********To my knowledge, NONE of these questions is more than partially answered. **********

Singularity of a bounded linear map $f : N \to M$, where M and N are any von Neumann algebras, was first introduced by Takesaki in 1958. A different definition was given by Akemann-Anderson in 1991, and a third definition was given by Pop in 1998. Let's clarify this situation.

DEFINITIONS: A bounded linear functional f on the von Neumann algebra M is called **singular** if every non-zero projection $p \in M$ dominates a non-zero projection $q \in M$, such that $f(q) = 0$. (Takesaki 1959) (The existence of singular functionals requires the Axiom of Choice.)

The dual space N^* of N can then be decomposed as $N^* = N_* \oplus N_*^{\perp}$, where N_* denotes the normal linear functionals of N and $N_*^{\perp}$ denotes the singular linear functionals of N.

A bounded linear map $T : M \to N$ from the von Neumann algebra M into the von Neumann algebra N is called **singular** if:

1. (Takesaki 1958) $T^*(N_*) \subset M_*^{\perp}$.
2. (Ak-Anderson 1991). $T^*(N^*) \subset M_*^{\perp}$.
3. (Pop 1998). Every non-zero projection $p \in M$ dominates a nonzero projection $q \in M$, such that $T(q) = 0$.

PROPOSITION 1.1: Using the definitions above, $3 \to 2 \to 1$. Further, $1 \to 3$ if N has a faithful normal state.

PROPOSITION 1.2: Using definitions 2 or 3, the composition of singular maps is singular. However, under definition 1, the composition of singular maps may be the identity map.

QUESTION: Are definitions 2 and 3 equivalent?

HOW DO WE FIND ELEMENTS OF S AND S_A ?

1. The Hahn-Banach Theorem as applied to locally convex spaces shows that f_0 has at least one normal state extension to N.

2. Krein's (1937) state extension procedure will produce at least one state extension.

3. If $E : N \to A$ is a projection of norm 1 (also called a conditional expectation, or CE), then $E^*(f_0)$ will lie in S_A. If E is a singular map, then $E^*(f_0)$ will be a singular extension of f_0. If E is normal, then $E^*(f_0)$ will be a normal extension of f_0.

4. If $f \in S \setminus S_A$, then there must be some projection $p \in A$ and some $b \in N$ such that $f(pb(1-p)) \neq 0$, i.e. $f \neq pfp + (1-p)f(1-p)$, and the RHS is also in S. This procedure suggests the following definition (that evolved through von Neumann, Kadison & Singer, and Arveson.

DEFINITION: Define a directed set W as follows. Elements of W will be finite subsets of projections in A with sum 1. Given two such subsets F, G, we say $F \geq G$ if F is a refinement of G (i.e. each element of F is dominated by an unique element of G). For $F \in W, g \in N^*, b \in N$ define $g_F \in N^*$ by $g_F = \sum_{p \in F}(pgp)$ and $b_F = \sum_{p \in F}(pbp)$. Note that this definition implies that $g(b_F) = g_F(b)$.

EASY LEMMA: Using the notation just above, There exists a subnet W' such that:

(i). $\lim_{F \in W'} b_F$ converges weak* for each $b \in N$ to a limit $E(b)$. E is a conditional expectation of N onto A.

(ii). $\lim_{F \in W'} g_F$ converges weak* for each $g \in N^*$ to a limit $\Phi(g)$ that has A in its centralizer and $g|_A = \Phi(g)|_A$.

ANOTHER METHOD FOR FINDING CEs

Theorem: If f is a state of N such that $f_0 = f|_A$ is faithful and normal and $f \in S_A$, then there is a conditional expectation $G : N \to A$ such that $G^*(f_0) = f$. Further, f is normal on N iff G is normal; and f is singular iff G is singular.

PROOF OUTLINE: Because f_0 is faithful and normal, it follows readily that $A(f_0)$ is weakly dense in A_*, hence norm dense by convexity of Af_0. Define a map $G_0 : Af_0 \to N^*$ by $G_0(af_0) = af$. We first show that G_0 is isometric and therefore can be extended to an isometry from A_* into N^*. If $a = u|a|(u^*a = |a|)$ is the polar decomposition, then

$||af_0|| \geq ||(af_0)u^*|| = ||a(f_0u^*)|| = ||a(u^*f_0)||$
$= ||(u^*a)f_0|| = |||a|f_0|| = |||a|^{1/2}f_0|a|^{1/2}||$
$= f_0(|a|^{1/2}1|a|^{1/2}) = f_0(|a|) = f(|a|) =$
$f(a|^{1/2}1|a|^{1/2}||) = |||a|f|| \geq ||(|a|f)u|| = |||a|(uf)|| =$
$||(u|a|f) = ||af|| \geq ||af_0||$ Thus all the inequalities are equalities, and isometry is proved.

Dualizing we get a norm 1 map $G_0^* : N^{**} \to A$. Identifying elements of N, including those in A, with their canonical images in N^{**}, we define $G = G_0^*|_N$. Under this identification, $G^*|_{A_*} = G_0$.

Next we show that G is a CE. For all $a, c \in A$
$(cf|_A)(G(a)) = G^*(cf_0))(a) = (G_0(cf_0)(a) =$

$$(cf)(a) = f(ca) = f_0(ca) = (cf_0)(a).$$

Since $A(f_0)$ is dense in $A_*, G(a) = a$ for all $a \in A$. The normality and singularity conclusions are straightforward.

The last theorem shows that:

COROLLARY 1: There are two distinct CEs from N onto A iff there exists normal state f_0 of A such that S_A is not a singleton.

This relationship sparked my initial interest in the problem of extending normal states from A to N.

As Kadison & Singer pointed out, those interested in extending pure states from A to N should also keep an eye on CEs. They noted that:

COROLLARY 2: If there are two distinct CEs from N onto A, then **some** pure state of A must fail to have unique state extension.

Comparing the two corollaries above, one is an "if-then" and the other is an "iff". I can't resist making the following:

CONJECTURE: The converse to Corollary 2 holds.

Since Kadison & Singer showed that, if $N = B(H)$ and A is a discrete MASA, then the "projection onto the diagonal" is the unique CE; proving the above Conjecture would also solve the Kadison & Singer problem by showing the uniqueness of pure sate extension in this case. Disproving the Conjecture would not solve the K&S problem, but might suggest a new line of attack.

QUESTION: If we use some methods to produce two elements of S or S_A (or two CEs) for some f_0, how do we know when these two elements are distinct?

Let's look now at a simple, yet instructive,
EXAMPLE. N is the algebra of 2x2 matrices and A is the algebra of diagonal matrices. If we choose to view elements of A and A_* in matrix form, then

$$a = \begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix}, f_0 = \begin{pmatrix} t_{11} & 0 \\ 0 & t_{22} \end{pmatrix}, f_0(a) = trace(af_0)$$

gives a typical state f_0 provided that t_{11}, t_{22} are nonnegative with sum 1. If z is any complex number with $|z| \leq \sqrt{t_{11} - t_{11}^2}$, then we can define a state extension

$$f_z \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = trace\left(\begin{pmatrix} t_{11} & z \\ \overline{z} & t_{22} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \right).$$

It is important to note that if $t_{11} = 0$ or $t_{11} = 1$, then

$$f_z = \begin{pmatrix} t_{11} & 0 \\ 0 & t_{22} \end{pmatrix}$$

because $z = 0$ is the only possible choice.

I.e. if f_0 is a pure state of A, then it has unique state extension, but any other state of A has many possible extensions, but S is still finite dimensional (2 real or one complex dimension) and hence compact. It is also clear that S_A is the singleton

$$\begin{pmatrix} t_{11} & 0 \\ 0 & t_{22} \end{pmatrix},$$

which is the center of the disk that comprises S

CONSIDER FAITHFUL STATES.

In order to escape from the possibility that f_0 might not "care about" most of A, it simplifies matters to assume that f_0 is faithful on A. After all, if p is the support projection of f_0 in A, and f is a state extension of f_0 to N, then f must vanish on $(1 - p)$, and hence $f = pfp$. Since pA is still a MASA in pNp and f_0 is now faithful on pA, assuming that f_0 is faithful results in no loss of generality.

COMMENT: If f_0 is faithful on A and f is a normal extension of f_0 to a state of N, f may not be faithful on N; indeed it may be singular. This possibility plays a key role in the type II_1 case a few slides down the road.

CONJECTURE: Given N, A, f_0, S as above, then S is a singleton iff $A = N$.

COMMENT: You are probably thinking, "That looks easy!" Yes, it looks that way to me also, but notice how the EXAMPLE relied on the abelian projections. Let's go to infinite dimensions and see how everything works out nicely, at least for discrete MASAs.

NOTATION: Let $N = B(H)$ denote a type I_n factor (n is a possibly infinite cardinal)) with discrete MASA A, f_0 a faithful, normal state of A, S the set of all state extensions of f_0 to N and S_A denote the set of elements $g \in S$ such that A is contained in centralizer of g. (Note that the assumption that f_0 is faithful requires that $n \leq \aleph_0$.) Because A is discrete, $f_0 = \sum_{k=1}^{n} t_k g_k$, where $\{g_k\}$ is a complete list of the normal pure states of A and $\sum_{k=1}^{n} t_k = 1$. Because f_0 is faithful, we must have $0 < t_j < 1 \quad \forall j = 1, ..., n$. Let $E : N \to A$ be the unique CE.

FACTS:

1. S is a singleton iff $n = 1$ (i.e. $N = A$).
2. S is finite dimensional iff n is finite.
3. S is a norm compact subset of N_*.
4. $E(f_0)$ is the unique element of S_A.

PROOF OUTLINE: $E(f_0)$ has exactly the same form as f_0, where the g_k now stand for the vector pure states along the diagonal. (See the 2x2 example earlier.) Since any normal $g \in S$ is uniquely represented by a trace class operator, that operator must be diagonal in order that A be in the centralizer of g. This gives 4.

1 and 2 will follow if I show that for any two non-zero t_k, say t_1, t_2, there is a positive extension of the positive functional $t_1 g_1 + t_2 g_2$. This was essentially done in the 2x2 example two slides ago.

If 3 is false, then S will contain non-normal state. However, since any singular state must vanish on minimal projections, this leads to an easy contradiction.

THEOREM: If N is semifinite, then the following are equivalent:

1. there exist abelian (for N) projections $\{p_t\} \subset A$ with $\sum p_t = 1$ (and in particular, N is type I);

2. no normal state of A has a singular state extension to N;

3. no normal state of A has a non-normal state extension to N.

PROOF OUTLINE: $3 \to 2$ is trivial.

$2 \to 3$. If 3 fails, then there is a normal state f_0 of A that has a non-normal state extension $f = f_n + f_s$. Then $f - f_n$ is normal and positive on A with a singular extension f_s. Therefore 2 fails.

$1 \to 2$. Suppose that 1 holds and that f is a singular state of N such that $f_0 = f|_A$ is normal. Since p_t is abelian, then $p_t N p \subset A$ for each t. Thus $f|_{p_t N p_t}$ is both normal and singular, hence equal to 0. But since f is normal on A and $\sum_t p_t = 1$, this means that $f = 0$, a contradiction.

$2 \to 1$. If 1 is false, then WLOG A contains no abelian projections at all. WLOG then there are two cases.

Case 1: Suppose A contains no finite projections. Let I be the closed ideal of N generated by the finite projections. The dual space of N/I is positively isometric to $I^\perp$, which by weak* density of I in N consists entirely of singular linear functionals. Since A contains no finite projections, A is isometrically imbedded in N/I, hence its dual space is the set of restrictions of functionals in $I^\perp$. We conclude that every normal state of A is the restriction of a singular state of N.

Case 2: If A contains a non-zero finite projection q, then qA is a MASA of qNq. We may assume that $q = 1$, so that N is finite. Now N cannot have a type I summand, because A would have nonzero abelian projections, contrary to assumption. So N is type II_1. Let τ be a normal, faithful (WLOG) tracial state of N. If $f_0 = \tau|_A$, we show that f_0 has a singular state extension.

For any n there are projections $\{q_j^n\}_{j=1}^{2^n} \subset A$ that are equivalent in N and have sum 1. For each i, j, n, let v_{ij}^n be a partial isometry effecting the equivalence of q_i^n and q_j^n, with the requirement that $v_{ii}^n = q_i^n$. Define $p_n = 2^{-n} \sum_{i,j=1}^{2^n} v_{ij}^n$. It is easy to check that p_n is a projection in N with $E(p_n) = 2^{-n}1$.

Define a sequence of states of N by $\varphi_n = 2^n \tau(\cdot p_n)$. Note that

$$\varphi_n(a) = 2^n \tau(ap_n) = 2^n \tau(E(ap_n))$$

$$= 2^n \tau(aE(p_n)) = 2^n \tau(a(2^{-n}1)) = \tau(a), \qquad a \in A,$$

so that any weak* subnet limit $\varphi = \lim_\alpha \varphi_{n_\alpha}$ of $\{\varphi_n\}$ in N^* extends $\tau|_A$. Moreover, for any m,

$$\varphi\left(\bigvee_{k=m}^{\infty} p_k\right) = \lim \varphi_{n_\alpha}\left(\bigvee_{k=m}^{\infty} p_k\right) \geq \lim \varphi_{n_\alpha}(p_{n_\alpha}) = 1,$$

$$\tau\left(\bigvee_{k=m}^{\infty} p_k\right) \leq \sum_{m}^{\infty} \tau(p_k) = 2^{-m+1}.$$

Considering the complements of the projections $\vee_{k=m}^{\infty} p_k$, we see that φ vanishes on projections of trace arbitrarily close to 1. Now given any projection $p \in N$, find another projection q with $\varphi(q) = 0$ and $\tau(p) + \tau(q) > 1$. The formula $p - (p \wedge q) \sim (p \vee q) - q$ implies $\tau(p \wedge q) = \tau(p) + \tau(q) - \tau(p \vee q) > 0$. Thus $0 \neq p \wedge q \leq p$ and $\varphi(p \wedge q) \leq \varphi(q) = 0$, as required to show that φ is singular.

QUESTION: Does the last result require the semifinite assumption, or is that just a convenience for the proof?

COROLLARY: If N is semi-finite and the support of f_0 dominates no abelian projection in A, then f_0 has a singular state extension.

PROOF OUTLINE: We only need to add one thing to the proof of the last theorem, namely that for the II_1 case, every normal state of A has a singular state extension.

Let f be a singular state extension of $\tau|_A$, let $a \in A^+$ with $\tau(a) = 1$, and define the state f_a on N by

$$f_a(b) = (a^{1/2} f a^{1/2})(b) = f(a^{1/2} b a^{1/2}).$$

Since the set of singular functionals of N is left and right invariant under multiplication by elements of N, then f_a is clearly singular. For $b \in A$,

$$f_a(b) = f(a^{1/2} b a^{1/2}) = f(ab) = \tau(ab) = (a\tau)(b).$$

But $\{a\tau : a \in A^+, \tau(a) = 1\}$ is norm dense in the normal states of A by the Radon-Nikodym Theorem.

Thus, for any normal state g of A there is a sequence $\{a_n\}$ of positive trace one elements of A such that $\|g - a_n\tau\|_1 \to 0$ (the norm of A_*). Each state $a_n\tau$ lifts to a singular state f_n, and let F' be the set of such lifts. Then F' is countable, so all its limit points in N^* will be singular states. Let K be the weak* closure of F' in N^*, so $K \subset N_s^*$. Consider the states of A determined by $K|_A$, i.e. restricting the states of K to A. Clearly $\{a_n\tau\} \subset K|_A$, and $K|_A$ is weak* closed, so $g \in K|_A$. Thus g has a singular state extension.

If we are willing to make the additional that A is "not too big", then we can tie the previous results to the existence of conditional expectations.

THEOREM: If A is singly-generated and N is semifinite, then the following are equivalent:

1. there exist abelian (for N) projections $\{p_t\} \subset A$ with $\sum p_t = 1$ (and in particular, N is type I);
2. no normal state of A has a singular state extension to N;
3. no normal state of A has a non-normal state extension to N;
4. there is a unique CE, $E : N \to A$, that is also normal and faithful;
5. there is a unique CE, $E : N \to A$.

COROLLARY: Suppose that N is semi-finite and A is singly generated. If the abelian (for N) projections in A don't have supremum equal to 1, then there are at least 2 CEs from N onto A, one of which is singular.

Let's review the situation. There are enough unsolved problems for a dissertation or three, but let me highlight some very specific conjectures.

1. If N is type III, every normal state f_0 of A has a singular extension.

COMMENT: It would be enough to show that there exists one normal state of A with a singular extension.

2. If $N = B(H)$ for a large H and if A is a continuous MASA of N, then there are at least two distinct CEs from N onto A.

COMMENT: I don't know what those MASAs look like.

3. If N is a type II_1 factor and $f_0 = \tau|_A$, S contains a pure state of N?

COMMENT: Take your favorite II_1 factor and MASA, and don't worry about generality; just prove **something** about normal state extensions and pure sates.

4. If N is a type II_1 factor and $f_0 = \tau|_A$, then S_A contains a singular state.

COMMENT: We know this if A is singly generated.