University of Colorado
Department of Mathematics

Problem of the Month

September 2021

Consider the $n \times n$ matrix

$$A_n = \left[a_{ij}\right] = \begin{bmatrix} 1 + \frac{(-1)^1}{1^2} \cdot \frac{(-1)^1}{1^2} & \frac{(-1)^1}{1^2} \cdot \frac{(-1)^2}{2^2} & \cdots & \frac{(-1)^1}{1^2} \cdot \frac{(-1)^n}{n^2} \\ \frac{(-1)^2}{2^2} \cdot \frac{(-1)^1}{1^2} & 1 + \frac{(-1)^2}{2^2} \cdot \frac{(-1)^2}{2^2} & \cdots & \frac{(-1)^2}{2^2} \cdot \frac{(-1)^n}{n^2} \\ \vdots & \vdots & & \vdots \\ \frac{(-1)^n}{n^2} \cdot \frac{(-1)^1}{1^2} & \frac{(-1)^n}{n^2} \cdot \frac{(-1)^2}{2^2} & \cdots & 1 + \frac{(-1)^n}{n^2} \cdot \frac{(-1)^n}{n^2} \end{bmatrix}$$

where

$$a_{ij} = \begin{cases} \frac{(-1)^i}{i^2}\frac{(-1)^j}{j^2} & \text{if } i \neq j \\ 1 + \frac{(-1)^i}{i^2}\frac{(-1)^j}{j^2} & \text{if } i = j. \end{cases}$$

Determine $\lim_{n\to\infty}(\det(A_n))$, and state whether this value is

(i) rational,
(ii) algebraic irrational, or
(iii) transcendental.