RETURN THIS COVER SHEET WITH YOUR EXAM AND SOLUTIONS!

Geometry/Topology

Ph.D. Preliminary Exam
Department of Mathematics
University of Colorado Boulder

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INSTRUCTIONS:

1. Answer each of the six questions on a separate page. Turn in a page for each problem even if you cannot do the problem.
2. Label each answer sheet with the problem number.
3. Put your number, not your name, in the upper right hand corner of each page. If you have not received a number, please choose one (1234 for instance) and notify the graduate secretary as to which number you have chosen.

Problem 1. Compute the fundamental group of $\mathbb{R}^3 - C$ where

$$C = \{(x, y, z) \mid x = 0, y^2 + z^2 = 2\}.$$

(Hint: Consider a tube around C whose inner hole has been filled by a disk.)

Problem 2. Let $p : X \to Y$ be a continuous closed surjection.

(a) Let $U \subset X$ be an open set which contains $p^{-1}(\{y\})$. Prove that there is an open neighborhood W_y of y such that $p^{-1}(W_y) \subset U$. (Hint: Consider $X \backslash U$.)

(b) Recall that X is normal if the one point sets in X are closed and, for every pair disjoint of closed sets A and B, there exists disjoint open sets U and V such that $A \subset U$ and $B \subset V$. Show that if X is normal, then so is Y.

Problem 3. Let $p : \widetilde{X} \to X$ be the universal cover of a connected and locally-path connected space X and let $A \subset X$ be a connected and locally path-connected subspace. Let $\widetilde{A}$ be a path component of $p^{-1}(A)$.

(a) Show that $\widetilde{A} \to A$ is a covering space.

(b) Prove that the image of

$$\pi_1\big(\widetilde{A}, \widetilde{a}_0\big) \to \pi_1(A, a_0)$$

coincides with the kernel of $\iota_* : \pi_1(A, a_0) \to \pi_1(X, a_0)$, where $\widetilde{a}_0 \in \widetilde{A}$ is any basepoint, $a_0 = p\big(\widetilde{a}_0\big)$, and $\iota : A \hookrightarrow X$ is the canonical embedding.

Problem 4. (a) Show that there is no immersion $S^1 \to \mathbb{R}$.

(b) Consider the function $f : \mathbb{R}^3 \to \mathbb{R}^2$, $(x, y, z) \mapsto (x^3 z, xy + z)$. At which point is f a submersion? Determine the regular values of f.

Problem 5. Define $\omega = dx_1 \wedge dx_2 + dx_3 \wedge dx_4 + dx_5 \wedge dx_6$ as a 2-form on $\mathbb{R}$. Show that no diffeomorphism $\varphi : \mathbb{R}^6 \to \mathbb{R}^6$ satisfying $\varphi^* \omega = \omega$ can map the unit sphere S^5 to a sphere of radius $r \neq 1$.
Hint: consider $\omega \wedge \omega \wedge \omega$.

Problem 6. Recall that an n-dimensional manifold M is called *parallelizable* if its tangent bundle is trivial. Which of the following manifolds are parallelizable? Provide a short justification of your answer in a sentence.

(i) The n-torus $\left(S^1\right)^n = \mathbb{R}^n / \mathbb{Z}^n$ (where $\mathbb{Z}^n$ is the subgroup of the additive group of $\mathbb{R}^n$ consisting of points whose coordinates are all integers);

(ii) the sphere S^2;

(iii) the real projective plane $\mathbb{RP}^2$.