RETURN THIS COVER SHEET WITH YOUR EXAM AND SOLUTIONS!

Geometry-Topology

Ph.D. Preliminary Examination

Department of Mathematics

University of Colorado

January, 2015

INSTRUCTIONS:

(1) Answer each of the six questions on a separate page. Turn in a page for each problem even if you cannot do the problem.
(2) Label each answer sheet with the problem number.
(3) **Put your number, not your name, in the upper right hand corner of each page**. If you have not received a number, please choose one (1234 for instance) and notify the graduate secretary as to which number you have chosen.

Problem 1. Let $(X, \mathfrak{T})$ be a topological space, and let $\sim$ be an equivalence relation defined on X. Denote by $(X/\sim, \mathfrak{Q})$ the set of equivalence classes with the quotient topology $\mathfrak{Q}$. Prove that if $(X/\sim, \mathfrak{Q})$ is Hausdorff, then every equivalence class C in X is a closed set of $(X, \mathfrak{T})$.

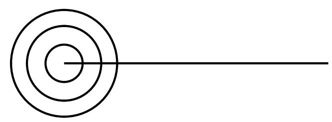
Problem 2. Let $Q = [0,1] \times [0,1]$ be the unit square with the topology induced from the standard topology in $\mathbb{R}^2$. Define the following equivalence relation $\sim$ on Q:

- every point is equivalent to itself;
- $(0, x)$ is equivalent to $(1, x)$ $\forall x \in [0,1]$;
- $(x, 0)$ is equivalent to $(1-x, 1)$ $\forall x \in [0,1]$.

The Klein bottle $\mathbb{KB}$ is defined to be the topological space $(Q/\sim, \mathfrak{Q})$ where $\mathfrak{Q}$ is the quotient topology.

(1) Show (pictorially if you like) that the 2-torus $\mathbb{T}^2$ is a double cover of the Klein bottle $\mathbb{KB}$.
(2) Can a closed orientable surface that is not homeomorphic to the 2-sphere or the 2-torus be a cover of the 2-torus $\mathbb{T}^2$? Explain.

Problem 3. Let $(\mathbb{R}^2, \mathcal{ST}_2)$ be $\mathbb{R}^2$ with the standard topology $\mathcal{ST}_2$. Let C be the subset of $\mathbb{R}^2$ defined below, and let $(C, \mathcal{ST}_2|_C)$ be C with the induced topology from $\mathcal{ST}_2$. Use the Seifert–Van Kampen Theorem to find the fundamental group of $(C, \mathcal{ST}_2|_C)$.



C

$$C := \left(S_1 \bigcup S_2 \bigcup S_3 \bigcup \{(x,0) | 0 \le x \le 5, x \in \mathbb{R}\} \right) \subset \mathbb{R}^2,$$

where

$$S_i = \{(x,y) \in \mathbb{R}^2 | x^2 + y^2 = i\}, \ i = 1,2,3.$$

Problem 4. Consider the vector fields on $\mathbb{R}^3$:

$$X = x(2y + \cos y)\frac{\partial}{\partial x} - \frac{\partial}{\partial z}, \quad Y = \frac{\partial}{\partial y} + \frac{\partial}{\partial z}.$$

(1) Write down a submanifold N of $\mathbb{R}^3$ containing the point $(0,1,0)$ and having the property that there is an open neighborhood $U \subseteq N$ of $(0,1,0)$ such that for each point $n \in U$ the tangent space $T_nN \subseteq T_n\mathbb{R}^3$ is equal to the span of $X(n)$ and $Y(n)$.
(2) Compute the Lie bracket $[X, Y]$.
(3) Let $\mathcal{D}$ be the set of vector fields that can be written in the form $fX + gY$ for some functions $f, g \in C^\infty(\mathbb{R}^3)$. Is $[X, Y]$ in $\mathcal{D}$? Explain.
(4) Can you write down a submanifold N of $\mathbb{R}^3$ containing the point $(1,0,0)$ and having the property that there is an open neighborhood $U \subseteq N$ of $(1,0,0)$ such that for each point $n \in U$ the tangent space $T_nN \subseteq T_n\mathbb{R}^3$ is equal to the span of $X(n)$ and $Y(n)$? Explain.

Problem 5. Let M be a smooth manifold.

(1) Let $p \in M$. Show there is an open neighborhood U of p such that for all $q \in U$, there is a diffeomorphism $f : M \to M$ such that $f(p) = q$. [Hint: you may want to consider flows of appropriately chosen vector fields, or, alternatively maps $\mathbb{R}^n \to \mathbb{R}^n$ of the form $x \mapsto x + \eta(x)y$ with $y \in \mathbb{R}^n$ and $\eta(x)$ an appropriately chosen function.]

(2) Deduce from part (1) that if M is connected, then the action of the diffeomorphism group of M on M is transitive; i.e., show that given any two points $p, q \in M$ there is a diffeomorphism $f : M \to M$ with $f(p) = q$.

Problem 6. Let (M, g) be a Riemannian manifold with Riemannian curvature tensor R. Consider the $(0, 2)$-tensor ω, defined by the trace

$$\omega(u, v) := \mathrm{tr}(w \mapsto R(u, v)(w)),$$

where u, v, w are vector fields on M. Show that $\omega = 0$.