RETURN THIS COVER SHEET WITH YOUR EXAM AND SOLUTIONS!

Geometry/Topology

Ph.D. Preliminary Exam
Department of Mathematics
University of Colorado Boulder

January, 2015

INSTRUCTIONS:

1. Answer each of the six questions on a separate page. Turn in a page for each problem even if you cannot do the problem.
2. Label each answer sheet with the problem number.
3. Put your number, not your name, in the upper right hand corner of each page. If you have not received a number, please choose one (1234 for instance) and notify the graduate secretary as to which number you have chosen.

Problem 1. Recall that a topological space X is said to be *normal* if its points are closed subsets and for every pair of disjoint closed subsets A and B of X there is a pair of disjoint open subsets U and V with $A \subset U$ and $B \subset V$. Let $\pi : X \to Y$ be a surjective continuous and closed map. Show that if X is a normal space, then Y is normal as well.

Problem 2. Let $X = S^1 \times S^1$. Prove that for every $n \geq 2$, every continuous map $S^n \to X$ is homotopic to a constant map.

Problem 3. Let $\mathrm{GL}_n(\mathbb{R})$ be the space of $n \times n$ invertible matrices with entries in $\mathbb{R}$ together with the natural manifold structure inherited from the ambient space of $n \times n$ matrics with entries in $\mathbb{R}$. Prove that the conjugation map

$$\mathrm{GL}_n(\mathbb{R}) \times \mathrm{GL}_n(\mathbb{R}) \to \mathrm{GL}_n(\mathbb{R}), \quad (g, h) \mapsto ghg^{-1}$$

is differentiable and compute its tangent map. Conclude that the conjugation map is continuous.

Problem 4. (i) Show that $\mathrm{GL}_n(\mathbb{R})$ with the topology as in Problem 3 is disconnected for all positive integers n.

(ii) Show that for even n the matrix $-1_n := \mathrm{diag}(-1, \cdots, -1)$ can be connected by a path in $\mathrm{GL}_n(\mathbb{R})$ with the identity matrix 1_n.

(iii) Let $\mathrm{GL}_n(\mathbb{R})^+$ be the set of matrices in $\mathrm{GL}_n(\mathbb{R})$ with positive determinant. Show that $\mathrm{GL}_n(\mathbb{R})^+$ is path connected. (Hint: Use Problem 3, Problem 4 (ii), and the fact that every matrix is conjugate to an upper triangular matrix to construct a path between a matrix with positive determinant and the identity matrix.)

(iv) Compute $H_0(\mathrm{GL}_n(\mathbb{R}))$ for all positive integers n.

Problem 5. Let M be a compact, oriented n-dimensional manifold (without boundary) and suppose that ω and η are p- and q-forms on M with $p + q = n - 1$. Prove that

$$\int_M d\omega \wedge \eta = (-1)^{p+1} \int_M \omega \wedge d\eta.$$

Problem 6. Let N be a compact embedded submanifold of a manifold M. Show that $\Omega^p(M) \to \Omega^p(N)$ is surjective for all p.