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Topology/Geometry

Ph.D. Preliminary Exam

January, 2014

INSTRUCTIONS:

1. Answer each question on a separate page. Turn in a page for each problem even if you cannot do the problem.

2. Label each answer sheet with the problem number.

3. Put your number, not your name, in the upper right hand corner of each page. If you have not received a number, please choose one (1234 for instance) and notify the graduate secretary as to which number you have chosen.

Topology/Geometry Prelim **January 2014**

1. Consider the topological spaces $Q_1 = S^1 \times \mathcal{B}^2 \subseteq R^4$ and $Q_2 = \mathcal{B}^2 \times S^1 \subseteq R^4$, where $\mathcal{B}^2$ is the unit disc in $\mathbb{R}^2$ and S^1 is its boundary, the unit circle. Endow Q_j with the topology induced from the standard topology on $\mathbb{R}^4$, $j = 1, 2$. Note in particular that $\partial Q_j = S^1 \times S^1$, $j = 1, 2$. Consider the quotient space X obtained by identifying $(w_1, w_2) \in Q_1$ with $(w_2, w_1) \in Q_2$ whenever w_1 and w_2 are both in the unit circle. Compute the fundamental group of X using the van Kampen theorem.

2. (a) Show that $\mathbb{R}$ is not homeomorphic to $\mathbb{R}^2$ (with the standard topologies).
   (b) Is the topological space $\mathbb{R}$ (endowed with the standard topology) homeomorphic to the topological space $\mathbb{R}$ (endowed with the finite complement topology)?

3. Let (M, d) be a metric space.

   (a) Show that the distance function $d : M \times M \to \mathbb{R}$ is continuous. Here $M \times M$ has the product topology.
   (b) If A and B are disjoint compact subsets of M, show that
   $$d(A, B) = \inf_{x \in A, y \in B} d(x, y)$$
   is positive, and that there are points $x_0 \in A$ and $y_0 \in B$ such that $d(A, B) = d(x_0, y_0)$.

4. Suppose M is a smooth orientable compact manifold of dimension $2n$. Suppose ω is a 2-form for which $d\omega = 0$. Let $\mu = \omega^n = \omega \wedge \omega \wedge \cdots \wedge \omega$ (n times), and suppose that μ is a nowhere-zero $2n$-form on M.

   (a) Show that $d\mu = 0$.
   (b) Show that μ is not $d\beta$ for any $(2n-1)$-form β.
   (c) Conclude that ω is not $d\alpha$ for any 1-form α.

5. Suppose $f \colon \mathbb{R}^2 \to \mathbb{R}^3$ is given by
   $$f(u, v) = (u + v, uv, u - v).$$
   (a) If $\omega = y\,dx + x\,dy + y\,dz$, compute $f^*\omega$.
   (b) Is $f^*\omega = d\beta$ for some 1-form β on $\mathbb{R}^2$?
   (c) Calculate $\int_{\partial I^2} f^*\omega$ over the boundary of the unit square I^2 in $\mathbb{R}^2$ defined by the region inside the lines $x = 0$, $x = 1$, $y = 0$, and $y = 1$.

6. Consider the map $F \colon \mathbb{R}^3 \to \mathbb{R}^2$ defined by
   $$F(u, v, w) = (w^2 - uv, v^2 + u^2).$$
   At which values (a, b) is $F^{-1}(a, b)$ a smooth one-dimensional submanifold of $\mathbb{R}^3$?