RETURN THIS COVER SHEET WITH YOUR EXAM AND SOLUTIONS!

Geometry/Topology

Ph.D. Preliminary Exam
Department of Mathematics
University of Colorado Boulder

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INSTRUCTIONS:

1. Answer each of the six questions on a separate page. Turn in a page for each problem even if you cannot do the problem.

2. Label each answer sheet with the problem number.

3. Put your number, not your name, in the upper right hand corner of each page. If you have not received a number, please choose one (1234 for instance) and notify the graduate secretary as to which number you have chosen.

Q.1 Let $\mathbb{C}^*$ be the multiplicative group of non-zero complex numbers. Let $\mathbb{C}^*$ act on $\mathbb{C}^{n+1} \setminus \{0\}$ by

$$t.(x_1, \ldots, x_{n+1}) = (tx_1, \ldots, tx_{n+1}).$$

Let $\mathbb{C}\mathrm{P}^n$ be the orbit space $(\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^*$ with the quotient topology. Show that $\mathbb{C}\mathrm{P}^n$ is compact.

Q.2 Prove that there is no homeomorphism between $\mathbb{R}^3$ and $\mathbb{R}^2$.

Q.3 Prove that any continuous map $\mathbb{R}\mathrm{P}^2 \to S^1 \times S^1$ is homotopic to a constant map.

Q.4 Let $M^{2\times 2}$ be the space of 2×2 matrices, and let $J = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Define a map $F : M^{2\times 2} \to M^{2\times 2}$ by

$$F(A) = A^T J A.$$

(a) At which matrices A does the differential $DF(A)$ have maximum rank?

(b) Show that the set

$$\{A \in M^{2\times 2} \mid A^T J A = A\}$$

is a submanifold of $M^{2\times 2}$.

Q.5 (a) Let M be an n-dimensional manifold. Prove that the tangent bundle TM is bundle-isomorphic to $M \times \mathbb{R}^n$ if and only if there exist n vector fields $X_1, \ldots, X_n$ on M such that the vectors $\{X_1(p), \ldots, X_n(p)\}$ are linearly independent for every $p \in M$.

(b) Let $\mathbb{T}^2 = S^1 \times S^1$ be the 2-torus. Use part (a) to show that the tangent bundle of $\mathbb{T}^2$ is isomorphic to $\mathbb{T}^2 \times \mathbb{R}^2$.

Q.6 (a) State Stokes' Theorem.

(b) Let M be the manifold consisting of $\mathbb{R}^3$ minus the z-axis, and consider the 1-form

$$\alpha = -\frac{y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy + (z+1)\,dz$$

on M.

i. Show that $d\alpha = 0$.

ii. Let C be the unit circle $x^2 + y^2 = 1$ in the xy-plane, and compute $\int_C \alpha$.

iii. Use Stokes' Theorem to show that C is not the boundary of any 2-chain in M.