RETURN THIS COVER SHEET WITH YOUR EXAM AND SOLUTIONS!

Geometry/Topology

Ph.D. Preliminary Exam
Department of Mathematics
University of Colorado Boulder

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INSTRUCTIONS:

1. Answer each of the six questions on a separate page. Turn in a page for each problem even if you cannot do the problem.

2. Label each answer sheet with the problem number.

3. Put your number, not your name, in the upper right hand corner of each page. If you have not received a number, please choose one (1234 for instance) and notify the graduate secretary as to which number you have chosen.

Q.1 Define a topology on the set $\mathbb{R}$ of real numbers by the condition that $U \subseteq \mathbb{R}$ is open if and only if it is either empty or contains the interval $[0, 1)$. Then

(a) What is the interior of the set $[0, 1]$? And its closure?

(b) Does this topology on $\mathbb{R}$ satisfy the T_0 condition?

(c) Is $\mathbb{R}$ connected in this topology?

(d) Is $\mathbb{R}$ compact in this topology?

Q.2 Consider the 2-torus $\mathbb{T}^2 = \mathbb{T} \times \mathbb{T}$, where $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$ is the unit circle.

(a) What is the universal cover of $\mathbb{T}^2$?

(b) Describe the one-point compactification of $\mathbb{T}^2$ minus two distinct points. What is the fundamental group of the one-point compactification of $\mathbb{T}^2$ minus two distinct points?

Q.3 Prove that the singular homology $H_t(X)$ of the space $X = pt$ consisting of a single point is equal to

$$H_t(X) = \begin{cases} \mathbb{Z} & \text{if } t = 0 \\ 0 & \text{if } t > 0 \end{cases}$$

Q.4 Let M be the subset of Euclidean $\mathbb{R}^3$ defined by the zeros of the function

$$f(x, y, z) = xy - z.$$

(a) Prove that M is a submanifold of $\mathbb{R}^3$.

(b) Define a local coordinate system on M and compute the Riemannian metric induced on M by its embedding into Euclidean $\mathbb{R}^3$ in terms of these local coordinates.

Q.5 Let G be a Lie group. A vector field $\mathbf{v}$ on G is *left-invariant* if, for all $g, h \in G$,

$$(L_g)_*(\mathbf{v}|_h) = \mathbf{v}|_{gh},$$

where L_g denotes left multiplication by g.

(a) Show that the space of left-invariant vector fields on G is isomorphic to the tangent space of G at the identity.

(b) Use part (a) to prove that the tangent bundle of a Lie group is trivial. (A one-sentence description of how to construct a trivialization of the tangent bundle of G is sufficient.)

(c) Show that the vector field

$$\mathbf{v} = x\frac{\partial}{\partial x}$$

on the (abelian) Lie group $(\mathbb{R}_{>0}, \times)$ (i.e., the group of positive real numbers under multiplication) is left-invariant, and compute its flow from an arbitrary point.

Q.6 (a) State Stokes' Theorem.

(b) Let M be a smooth manifold, and $\omega \in \Omega^r(M)$ an r-form on M. Suppose that $\int_\Sigma \omega = 0$ for every r-dimensional submanifold Σ of M which is diffeomorphic to an r-sphere. Prove that $d\omega = 0$.