RETURN THIS COVER SHEET WITH YOUR EXAM AND SOLUTIONS!

Geometry-Topology

Ph.D. Preliminary Examination

Department of Mathematics

University of Colorado

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INSTRUCTIONS:

(1) Answer each of the six questions on a separate page. Turn in a page for each problem even if you cannot do the problem.
(2) Label each answer sheet with the problem number.
(3) **Put your number, not your name, in the upper right hand corner of each page**. If you have not received a number, please choose one (1234 for instance) and notify the graduate secretary as to which number you have chosen.

Problem 1. For $j = 0, 1$, let $T_j \subset \mathbb{R}^3$ be the 2-torus obtained by rotating the circle $C_j \subset \mathbb{R}^3$

$$C_j = \begin{cases} x = 0 \\ (y-3)^2 + (z-3j)^2 = 1 \end{cases}$$

about the z-axis; endow T_j with the subspace topology induced from the standard topology on $\mathbb{R}^3$. Parameterize the circle C_j by:

$$c_j(t) = (0, 3+\cos(t), 3j + \sin(t)), \; t \in [0, 2\pi], \; j = 0, 1.$$

Let $X := \Big(T_0 \sqcup T_1\Big)/\sim$ be the quotient space of the disjoint union

$$T_0 \sqcup T_1$$

modulo the equivalence relation $\sim$ generated by:

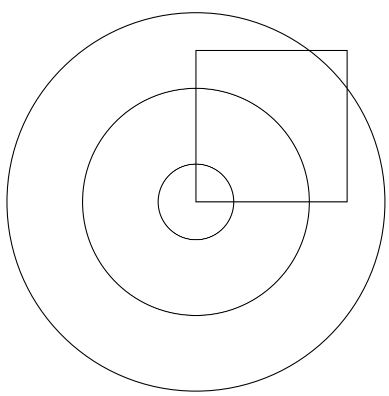
$$\forall t \in [0, 2\pi] : \; c_0(t) \sim c_1(t).$$

(1) What is $H_0(X)$? Explain.
(2) Use the Seifert–van Kampen Theorem to compute the fundamental group of X.

Problem 2. Let $(\mathcal{X}, \mathfrak{T})$ be a topological space, and let $\{A_\lambda : \lambda \in \Lambda\}$ be a collection of compact subsets of $\mathcal{X}$.

(1) Prove that if Λ is finite, then $\bigcup_{\lambda \in \Lambda} A_\lambda$ is compact.
(2) Give an example of a topological space $(\mathcal{X}, \mathfrak{T})$, an infinite set Λ, and a collection $\{A_\lambda : \lambda \in \Lambda\}$ of pairwise disjoint non-empty compact subsets of $\mathcal{X}$ such that $\bigcup_{\lambda \in \Lambda} A_\lambda$ is compact.

Problem 3. Consider the topological subspace Y of $\mathbb{R}^2$, with the subspace topology induced from the standard topology on $\mathbb{R}^2$, and formed by the lines as in the figure below:



(1) The space Y is homotopic to a bouquet of circles. How many? Explain.
(2) Is the space Y homeomorphic to a bouquet of circles? Explain.

Problem 4. Let n be a positive integer, and let (a_{ij}) be a nonzero symmetric $(n+1) \times (n+1)$ matrix with real entries a_{ij}. Consider the function

$$f : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$$

$$f(x_0, \ldots, x_n) = \sum_{i,j=0}^{n} a_{ij} x_i x_j,$$

and let $Q := f^{-1}(0) \cap (\mathbb{R}^{n+1} - \{0\})$ be the corresponding zero set of f in $\mathbb{R}^{n+1} - \{0\}$.

(1) Use the Implicit Function Theorem to show that Q is a smooth submanifold of $\mathbb{R}^{n+1} - \{0\}$, provided that $\det(a_{ij}) \neq 0$, i.e., the determinant of the matrix (a_{ij}) is nonzero.

(2) Can the converse to part (1) fail? In other words, can Q be a smooth submanifold of $\mathbb{R}^{n+1} - \{0\}$ even if $\det(a_{ij}) = 0$? Explain.

Problem 5. Let M be a smooth manifold. Denote by $\mathfrak{X}(M)$ the vector space of smooth vector fields on M, and for each nonnegative integer k denote by $\Omega^k(M)$ the vector space of smooth k-forms on M.

(1) Let $X_1, \ldots, X_{k+1} \in \mathfrak{X}(M)$ and $\omega \in \Omega^k(M)$. It is a fact that

$$\begin{aligned} d\omega(X_1, \ldots, X_{k+1}) = &\sum_{i=1}^{k+1} (-1)^{i+1} X_i(\omega(X_1, \ldots, \hat{X}_i, \ldots, X_{k+1})) \\ &+ \sum_{1 \le i < j \le k+1} (-1)^{i+j} \omega([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{k+1}). \end{aligned}$$

Confirm this fact for 1-forms; i.e., for the case $k = 1$. The notation $\hat{X}_i$ indicates that X_i is to be omitted.

(2) Given a covariant derivative ∇ on M, there is an induced map

$$D : \mathfrak{X}(M) \times \Omega^k(M) \longrightarrow \Omega^k(M)$$

$$\begin{aligned} (D_X\omega)(X_1, \ldots, X_k) = &X\left(\omega(X_1, \ldots, X_k)\right) \\ &- \sum_{i=1}^{k} \omega\left(X_1, \ldots, X_{i-1}, \nabla_X X_i, X_{i+1}, \ldots, X_k\right) \end{aligned}$$

where $X, X_1, \ldots, X_k \in \mathfrak{X}(M)$ and $\omega \in \Omega^k(M)$; here we are denoting $D(X, \omega) = D_X\omega$. It is a fact that for $X_1, \ldots, X_{k+1} \in \mathfrak{X}(M)$ and $\omega \in \Omega^k(M)$, if ∇ is torsion-free, then

$$d\omega(X_1, \ldots, X_{k+1}) = \sum_{i=1}^{k+1} (-1)^{i+1} (D_{X_i}\omega)(X_1, \ldots, \hat{X}_i, \ldots, X_{k+1}).$$

Confirm this fact for 1-forms; i.e., for the case $k = 1$. [Hint: Use part (1), and recall that the torsion T of ∇ is defined by $T(X_1, X_2) = \nabla_{X_1} X_2 - \nabla_{X_2} X_1 - [X_1, X_2]$.]

(3) Given a covariant derivative ∇ on M, we say that a smooth k-form $\omega \in \Omega^k(M)$ is parallel with respect to ∇ if $D_X\omega = 0$ for all smooth vector fields $X \in \mathfrak{X}(M)$. Use part (2) of this problem (without proof) to show that if a smooth k-form $\omega \in \Omega^k(M)$ is parallel with respect to a torsion-free connection ∇, then ω is d-closed.

Problem 6. Let M be a compact oriented manifold of dimension n without boundary, and let k be an integer such that $0 \le k \le n$. Show that if $\omega \in \Omega^k(M)$ is d-exact and $\eta \in \Omega^{n-k}(M)$ is d-closed, then one has

$$\int_M \omega \wedge \eta = 0.$$