RETURN THIS COVER SHEET WITH YOUR EXAM AND SOLUTIONS!

Geometry/Topology

Ph.D. Preliminary Exam
Department of Mathematics
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INSTRUCTIONS:

1. Answer each of the six questions on a separate page. Turn in a page for each problem even if you cannot do the problem.

2. Label each answer sheet with the problem number.

3. Put your number, not your name, in the upper right hand corner of each page. If you have not received a number, please choose one (1234 for instance) and notify the graduate secretary as to which number you have chosen.

Q.1 Suppose that X is a topological space, and let

$$\Delta = \{(x,x) \mid x \in X\}.$$

Prove that X is Hausdorff if and only if Δ is a closed subset of $X \times X$.

Q.2 For each n, let $X_n \subset \mathbb{R}^2$ be the circle of radius $\frac{1}{n}$ centered at $(\frac{1}{n}, 0)$. Let $X = \bigcup_{n=1}^{\infty} X_n$. Prove that X has no universal cover.

Q.3 Suppose that E is a contractible topological space and G is a group acting freely and properly discontinuously on E. Let e be a point of E, let $X = E/G$ be the set of G-orbits, with the quotient topology, and let x be the image of e in X.

Recall that $\mathbb{R}P^2$ is the quotient of the 2-sphere S^2 by the equivalence relation $(x,y,z) \sim (-x,-y,-z)$, with the quotient topology. With X as above, prove that there is a continuous map $\mathbb{R}P^2 \to X$ that is not homotopic to a constant if and only if there is an element $g \in G$ such that $g \neq 1$ but $g^2 = 1$. (Hint: What is $\pi_1(X,x)$?)

Q.4 Consider the subset $S \subset \mathbb{R}^3$ defined by the equations

$$x^2 + y^2 = a, \qquad yz = b,$$

where a, b are real numbers with $a > 0$.

(a) Show that if $b \neq 0$, then S is a smooth submanifold of $\mathbb{R}^3$.

(b) Show that S is not a smooth submanifold of $\mathbb{R}^3$ when $b = 0$.

Q.5 Consider the two vector fields on $\mathbb{R}^2$ given by

$$X = x\frac{\partial}{\partial x} - 2y\frac{\partial}{\partial y}, \qquad Y = \frac{\partial}{\partial y}.$$

(a) Find the smooth flow on $\mathbb{R}^2$ whose infinitesimal generator is X; i.e., find the (unique!) smooth map $\theta_t : \mathbb{R}^2 \to \mathbb{R}^2$ with the property that $\theta_0 = \mathrm{Id}\,|_{\mathbb{R}^2}$ and

$$\left.\frac{d}{dt}\right|_{t=0} \theta_t(p) = X_p, \qquad \text{for all } \ p \in \mathbb{R}^2.$$

(b) Find $\mathcal{L}_X Y$ using part (a) and the definition of Lie derivative

$$\mathcal{L}_X Y = \left.\frac{d}{dt}\right|_{t=0} d(\theta_{-t})(Y).$$

(c) Compute the Lie bracket $[X,Y]$ directly and check that your answer is the same as your answer to part (b).

Q.6 Let M be a smooth, oriented, $2n$-dimensional manifold. A 2-form ω on M is called a *symplectic form* on M if $d\omega = 0$ and the $2n$-form $\omega^n = \overbrace{\omega \wedge \cdots \wedge \omega}^{n \text{ times}}$ is a nowhere-vanishing $2n$-form on M. (This means that, in terms of any local coordinate chart $(U, (x^1, \ldots, x^{2n}))$ on M, we can write $\omega^n = f dx^1 \wedge \cdots \wedge dx^{2n}$ for some nonvanishing function $f : U \to \mathbb{R}$.)

(a) Let ω be a symplectic form on M. Let $(U, \mathbf{x})$ and $(V, \mathbf{y})$ be local coordinate charts on M that are compatible with the orientation of M, and suppose that $U \cap V \neq \emptyset$. Show that on $U \cap V$, when we write ω^n as

$$\omega^n = f\, dx^1 \wedge \cdots \wedge dx^{2n} = g\, dy^1 \wedge \cdots \wedge dy^{2n},$$

the nonvanishing functions $f, g : U \cap V \to \mathbb{R}$ must have the same sign; i.e., they are either both positive-valued or both negative-valued.

(b) Suppose that $M = \mathbb{R}^4$. Show that the 2-form

$$\omega = dx^1 \wedge dx^2 + dx^3 \wedge dx^4$$

is an exact symplectic form on $\mathbb{R}^4$. (Recall that a 2-form ω on M is *exact* if $\omega = d\alpha$ for some 1-form α on M.)

(c) Suppose that M is compact with no boundary. Show that no symplectic form on M is exact.