Department of Mathematics, University of Colorado Boulder

Topology/Geometry Preliminary Examination

August 2013

The six problems have equal points. Please do all of them.

1. Suppose C_k is a nested sequence of closed subsets of a compact space X; that is, $C_{k+1} \subset C_k$ for each $k \in \mathbb{N}$. Let $C = \cap_{k=1}^{\infty} C_k$. If U is an open set such that $C \subset U$, show that $C_k \subset U$ for some k.

2. Let $\mathbb{T}^2$ denote the quotient space $\mathbb{R}^2/\mathbb{Z}^2$, and let M denote the quotient of $\mathbb{T}^2$ by the relation $(x, y) \equiv (-x, -y)$.
   (a) Is the quotient $Q \colon \mathbb{T}^2 \to M$ a covering map?
   (b) Express M as a quotient of a polygon with sides identified.
   (c) What is the fundamental group of M?

3. State and prove Brouwer's Fixed Point Theorem for the closed unit disc D^2.

4. Let M be a C^∞ manifold.
   (a) Define *orientability* of M.
   (b) Construct coordinate charts for the tangent bundle TM.
   (c) Show that the tangent bundle TM of a smooth manifold M is always orientable, even if M itself is not.

5. Let $f \colon \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x, y) = x^3 + xy + y^3$.
   (a) Show that $f^{-1}(1)$ is a smooth submanifold of $\mathbb{R}^2$.
   (b) Show that $f^{-1}(0)$ is not a smooth submanifold. (Hint: if $(x(t), y(t))$ is a curve in $f^{-1}(0)$ with $x(0) = 0$ and $y(0) = 0$, what is the condition on $x'(0)$ and $y'(0)$?) In fact you can show that $f^{-1}(0)$ is not even a topological submanifold.

6. Let M be a compact, connected, and orientable smooth manifold of dimension 6. Let α and β be two 2-forms on M. Show that there is a point of M where $d\alpha \wedge d\beta = 0$. Hint: integrate.