GEOMETRY/TOPOLOGY PRELIMINARY EXAM
AUGUST 2011

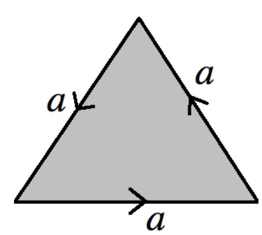
1. Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be a continuous function. Define an equivalence relation on $\mathbb{R}^2$ by $x \sim y$ if and only if $f(x) = f(y)$. Let X be the quotient space.

   (a) Show that X is always Hausdorff.

   (b) Must X be connected?

2. Let D^2 denote the unit disc in $\mathbb{R}^2$ with the unit circle S^1 its boundary. If $f\colon D^2 \to D^2$ is a homeomorphism, show that the restriction $f|_{S^1}$ is a homeomorphism onto S^1. (Hint: one way to do this is to assume it is not and obtain a contradiction by considering fundamental groups.)

3. Consider the quotient space Q formed by identifying the sides of a triangle T as in the diagram.



   (a) Is Q a topological manifold? (An intuitive explanation is sufficient.)

   (b) Use the Seifert-van Kampen theorem to compute the fundamental group of Q.

4. A *contact form* on a three-dimensional manifold M is a C^∞ 1-form on M such that $\alpha \wedge d\alpha$ is nowhere zero. A *Reeb field* for a contact form is a C^∞ vector field X on M such that $\alpha(X) = 1$ everywhere and $d\alpha(X, Y) = 0$ everywhere for every C^∞ vector field Y on M.

   (a) Prove that $\alpha = (\cos z)\, dx + (\sin z)\, dy$ is a contact form on $\mathbb{T}^3 = (\mathbb{R}/2\pi\mathbb{Z})^3$.

   (b) Show that there is a unique Reeb field for this contact form, and compute it.

(c) Describe the flow of this Reeb field. Is it periodic?

5. Let a and b be real numbers with $a > 0$. Consider the set M_{ab} of 2×2 matrices $A = \begin{pmatrix} w & x \\ y & z \end{pmatrix}$ satisfying $w^2+x^2+y^2+z^2 = a$ and $wz - xy = b$. Show that if $a \neq 2|b|$, then M_{ab} is a smooth submanifold of $\mathbb{R}^4$.

6. Suppose M is an annulus $[a, b] \times S^1$, for numbers $b > a > 0$, with C^∞ Riemannian metric given in polar coordinates $r \in [a, b]$ and $\theta \in S^1$ by $ds^2 = dr^2 + \varphi(r)^2 \, d\theta^2$ for some function φ. Let ∇ denote the usual Levi-Civita covariant derivative, and let R denote the radial vector field $R = r \frac{\partial}{\partial r}$.

   (a) Find all vector fields of the form $V = f(r) \frac{\partial}{\partial r} + g(r) \frac{\partial}{\partial \theta}$ satisfying $\nabla_R V = 0$ everywhere on M.

   (b) How does your answer change if M is a disc rather than an annulus, with the same metric? (Hint: what does $\varphi(r)$ look like near the origin $r = 0$?)