*RETURN THIS COVER SHEET WITH YOUR EXAM AND SOLUTIONS!*

Analysis

**Ph.D.**

**Preliminary Exam**

January, 2017

**INSTRUCTIONS:**

(1) Answer each of the six questions on a separate page. Turn in a page for each problem even if you cannot do the problem.
(2) Label each answer sheet with the problem number.
(3) Put your number, **not your name**, in the upper right hand corner of each page. If you have not received a number, please choose one (1234 for instance) and notify the graduate secretary as to which number you have chosen.

GOOD LUCK!

**Problem A.** Let $\Gamma$ be the 'curve'

$$\{(x, y) \in \mathbb{R}^2 \mid y = f(x)\},$$

where $f$ is a continuous function on the real line. Show that $m(\Gamma) = 0$, where $m$ is two-dimensional Lebesgue-measure (area). **Hint:** cover $\Gamma$ by rectangles and use uniform continuity.

**Problem B.**

Let $E$ be a given set $\mathbb{R}^2$ and let $d$ denote the distance between a point and the set:

$$d(x,E) := \inf\{\sqrt{(x_1-y_1)^2+(x_2-y_2)^2},\ y \in E\}.$$

(Here $x=(x_1,x_2)$ and $y=(y_1,y_2)$.)

Consider the open set

$$O_n := \{x \mid d(x,E) < 1/n\}.$$

Show that

(1) If $E$ is compact then $m(E) = \lim_{n\to\infty} m(O_n)$, where $m$ is two-dimensional Lebesgue-measure.

(2) If $E$ is closed and unbounded or $E$ is open and bounded, then the conclusion is false. **Hint:** In the second case, consider a dense countable set in the unit square.

**Problem C.**

Prove that in a Hilbert space, if $|\langle f, g\rangle| = \|f\| \, \|g\|$ and $g \neq \mathbf{0}$, then $f = cg$ for some scalar $c$. **Hint:** Consider the normalized vectors and use the Pythagoras theorem.

**Problem D.**

Let $f\colon \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be continuous, and such that the Lebesgue integral

$$\int_{\mathbb{R}} f(x,s)\,dx$$

exists for all $s\in\mathbb{R}$. Assume that, for any $x\in\mathbb{R}$, $f(x,s)$ is differentiable everywhere with respect to $s$, and that, for some function $g\in L^1(\mathbb{R})$,

$$\left|\frac{\partial}{\partial s}f(x,s)\right|\le g(x)$$

for all $(x,s)\in\mathbb{R}\times\mathbb{R}$. Show that you can differentiate under the integral sign: that is, show that

$$\frac{d}{ds}\int_{\mathbb{R}} f(x,s)\,dx=\int_{\mathbb{R}}\frac{\partial}{\partial s}f(x,s)\,dx.$$

Do not use any theorems you may already know about differentiating under the integral sign. However, you may use results such as the Dominated Convergence Theorem or the Bounded Convergence Theorem. Also, the Mean Value Theorem may be of help.

**Problem E.** Let
$$h(x,y) := \frac{xy}{(x^2+y^2)^2}.$$

(1) Explain why the integrals
$$\int_{\mathbb{R}} h(x,y)\,dx \quad \text{and} \quad \int_{\mathbb{R}} h(x,y)\,dy$$
exist (in the Lebesgue sense) for any $y$ or for any $x$ respectively, and evaluate these integrals.

(2) Explain why the integrals
$$\int_{\mathbb{R}}\left(\int_{\mathbb{R}} h(x,y)\,dx\right)dy \quad \text{and} \quad \int_{\mathbb{R}}\left(\int_{\mathbb{R}} h(x,y)\,dy\right)dx$$
exist (in the Lebesgue sense), and evaluate these integrals.

(3) Show that
$$\int_{\mathbb{R}^2} h(a)\,da,$$
where $a \in \mathbb{R}^2$ and $da$ is the usual Lebesgue measure on $\mathbb{R}^2$, does not exist.

**Problem F.**

Show carefully (that is, with any necessary convergence arguments) that, if $\{\sigma_n \colon n \in \mathbb{Z}^+\}$ is an orthonormal basis for a Hilbert space $H$, with respect to an inner product $\langle \cdot, \cdot \rangle$ on $H$, then for any $f, g \in H$ we have

$$\langle f, g \rangle = \sum_{n=1}^{\infty} \langle f, \sigma_n \rangle \langle \sigma_n, g \rangle$$

(meaning that the partial sums of the series on the right converge to the quantity on the left).