RETURN THIS COVER SHEET WITH YOUR EXAM AND SOLUTIONS!

Analysis

Ph.D. Preliminary Exam
Department of Mathematics
University of Colorado Boulder

January, 2016

INSTRUCTIONS:

1. Answer each of the six questions on a separate page. Turn in a page for each problem even if you cannot do the problem.

2. Label each answer sheet with the problem number.

3. Put your number, not your name, in the upper right hand corner of each page. If you have not received a number, please choose one (1234 for instance) and notify the graduate secretary as to which number you have chosen.

1. If (X, Σ, μ) is a measure space and if f is μ integrable, show that for every $\varepsilon > 0$ there is $E \in \Sigma$ such that $\mu(E) < \infty$ and

$$\int_{X \setminus E} |f| \, d\mu < \varepsilon$$

2. Let $\{f_n\}$ be a sequence of measurable functions on $[0,1]$, and suppose that for every $a > 0$ the infinite series $\sum\limits_{n=1}^{\infty} \mu\left(\{x \in [0,1] \mid |f_n(x)| > a\}\right)$ converges; here μ is the Lebesgue measure. Prove that

$$\lim_{n\to\infty} f_n(x) = 0$$

for almost every $x \in [0,1]$.

3. Let $f\colon [a,b] \to \mathbb{R}$ be a continuous function.

(a) Let $h > 0$. Show that the function $g_h(x) = \sup_{0<t<h} \frac{f(x+t)-f(x)}{t}$ is measurable;

(b) Show that $g(x) = \limsup_{t\to 0+} \frac{f(x+t)-f(x)}{t}$ is measurable;

(c) Prove that the set of points where f is differentiable is measurable.

4. Let f be integrable on the real line with respect to the Lebesgue measure. Evaluate

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f(x-n) \left(\frac{x}{1+|x|} \right) dx.$$

Justify all steps.

5. Let f be a non-negative measurable function on $(-\infty, \infty)$ such that $f(x) < \infty$ μ-almost everywhere; here μ is the Lebesgue measure. Prove or give a counterexample to each of the following:

a) For every N there exists a compact K such that $\mu(K) > N$ and f is integrable over K

b) There exist $a < b$ such that f is integrable over $[a,b]$.

6. A C^∞ function $f : \mathbb{R} \to \mathbb{R}$ satisfies the condition

for each $x \in \mathbb{R}$ there exists $n_x \in \mathbb{N}$ such that $f^{(n_x)}(x) = 0$.

Prove that f is a polynomial.