RETURN THIS COVER SHEET WITH YOUR EXAM AND SOLUTIONS!

Analysis

Ph.D. Preliminary Exam

January, 2014

INSTRUCTIONS:

1. Answer each question on a separate page. Turn in a page for each problem even if you cannot do the problem.

2. Label each answer sheet with the problem number.

3. Put your number, not your name, in the upper right hand corner of each page. If you have not received a number, please choose one (1234 for instance) and notify the graduate secretary as to which number you have chosen.

Analysis Prelim **January 2014**

1. Let X be a metric space, $A \subset X$ a compact subset and $p \in X \setminus A$ a point of X not in A. Prove that there exist disjoint open sets O_1 and O_2 in X such that $A \subset O_1$ and $p \in O_2$.

2. Let $f(x)$ be a continuous real-valued function on $[0,1]$ which satisfies

$$\int_0^1 f(x)x^n dx = 0 \text{ for } n = 0, 1, 2, \ldots.$$

   Prove that $f(x)$ is identically 0.

   Hint: You may find the (Stone-)Weierstrass theorem useful.

3. Let f, g be nonnegative, measurable functions on $[0,1]$ such that

$$\int_0^1 f(x)dx = 2, \quad \int_0^1 g(x)dx = 1, \quad \int_0^1 f(x)^2 dx = 5.$$

   Let $E = \{x \in [0,1] \mid f(x) \geq g(x)\}$. Show that $m(E) \geq 1/5$ (m is the Lebesgue measure).

4. Assume that $f\colon [0,1] \to \mathbb{R}$ is an absolutely continuous function with $\int_0^1 f(x)dx = 0$. Prove for any $y \in [0,1]$ that

$$\left| \int_0^1 (y-x)f'(x)dx \right| \leq \sup_{0 \leq x \leq 1} |f(x)|$$

5. Let $f \in L^3[-1,1]$. Show that

$$\int_{-1}^1 \frac{|f(x)|}{\sqrt{|x|}} dx < \infty$$

6. (a) Show that for $x > 0$ the limit $\lim_{R\to\infty} \int_0^R \frac{\cos t}{x+t} dt$ exists.

   (b) Define for $x > 0$

$$f(x) = \lim_{R\to\infty} \int_0^R \frac{\cos t}{x+t} dt.$$

   Show that $f(x)$ is continuous on $(0, \infty)$.