RETURN THIS COVER SHEET WITH YOUR EXAM AND SOLUTIONS!

Analysis

Ph.D. Preliminary Exam

January, 2011

INSTRUCTIONS:

1. Answer each of the six questions on a separate page. Turn in a page for each problem even if you cannot do the problem.

2. Label each answer sheet with the problem number.

3. Put your number, not your name, in the upper right hand corner of each page. If you have not received a number, please choose one (1234 for instance) and notify the graduate secretary as to which number you have chosen.

Analysis Prelim **January, 2011**

1. Let $\{f_n\}$ be a sequence of measurable real-valued functions on $[0,1]$. Show that the set of x for which $\lim_{n\to\infty} f_n(x)$ exists is measurable.

2. Let $\{f_n\}$ be a sequence of measurable functions on a $[0,1]$, and suppose that $\sum_{n=1}^{\infty} m(\{x \in [0,1] \mid f_n(x) > 1\}) < \infty$ where m is Lebesgue measure on [0,1]. Prove that $\limsup f_n(x) \le 1$ for almost every $x \in [0,1]$.

3.

(a) Let f be a real-valued Lebesgue measurable function defined on $[0,1]$. Give the definition of the essential supremum of f, $\|f\|_\infty$, and prove that if f and g are real-valued functions defined on $[0,1]$ whose essential supremums are finite, then $f+g$ is defined for almost all $x \in [0,1]$.

(b) Let $f : [0,1] \to \mathbb{R}$ be a Lebesgue measurable function with $\|f\|_\infty < \infty$. Prove that

$$\|f\|_\infty = \sup\left\{ \left|\int_{[0,1]} f(x)g(x)dx\right| : \ g \in L^1[0,1],\ \|g\|_1 = 1 \right\}.$$

4. Suppose that $\{f_n\}_{n=1}^{\infty} \in L^\infty[a,b]$, where $-\infty < a < b < \infty$. Let $f \in L^1[a,b]$.

(a) Show that for all $n \ge 1$, $f_n \in L^1[a,b]$.

(b) If $f_n \to f$ in $L^1[a,b]$, and $\sup_{n\ge1}\|f_n\|_\infty < \infty$, prove that $f \in L^\infty[a,b]$.

(c) Assuming part (b), prove that for all $p \in (1,\infty)$, $f_n \to f \in L^p[a,b]$.

5.

(a) Prove that for every $x > 0$, $\frac{1}{x} = \int_0^\infty e^{-xt} dt$.

(b) Prove that

$$\frac{\partial}{\partial x}\left[\frac{e^{-xt}(-t\sin x - \cos x)}{t^2+1}\right] = e^{-xt}\sin x.$$

(c) Using parts (a) and (b), prove that

$$\lim_{A\to\infty}\int_0^A \frac{\sin x}{x}dx = \frac{\pi}{2}.$$

State any theorems that you are using in your proof.

6. Let f_n be a sequence of real valued C^1 functions on $[0,1]$ such that, for all n,

$$|f_n'(x)| \le \frac{1}{\sqrt{x}} \text{ for } x > 0,$$
$$\int_0^1 f_n(x)dx = 0.$$

Prove that the sequence has a subsequence that converges uniformly on $[0,1]$.