RETURN THIS COVER SHEET WITH YOUR EXAM AND SOLUTIONS!

Analysis

Ph.D. Preliminary Exam

January, 2010

INSTRUCTIONS:

1. Answer each of the six questions on a separate page. Turn in a page for each problem even if you cannot do the problem.

2. Label each answer sheet with the problem number.

3. Put your number, not your name, in the upper right hand corner of each page. If you have not received a number, please choose one (1234 for instance) and notify the graduate secretary as to which number you have chosen.

Analysis Prelim **January, 2010**

1. Prove or disprove the following statement: If the real-valued function f is continuous on $[a, \infty)$ and $\int_a^\infty f(x)\,dx$ is convergent, then $\lim_{x\to\infty} f(x) = 0$.

2. Let (X, d) be a separable metric space. Suppose $S \subset X$. Show that there exists a countable set $F \subset S$ such that F is dense in S, i.e., such that $S \subset \overline{F}$.

3. Let $1 < p < 2$.

(a) Give an example of a function $f \in L^1(\mathbb{R})$ such that $f \notin L^p(\mathbb{R})$ and a function $g \in L^2(\mathbb{R})$ such that $g \notin L^p(\mathbb{R})$.

(b) Prove that if $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, then $f \in L^p(\mathbb{R})$.

4. Let f be continuously differentiable on $[a, b]$ with $f(a) = 0$. Let $M := \sup_{a\le x\le b} |f(x)|$. Show that

$$M^2 \le (b-a)\int_a^b f'^2(x)\,dx.$$

5. Let $f \in \mathcal{C}^2\big([-1,1]\big)$, that is, suppose f is twice continuously differentiable on $[-1, 1]$. Prove that

$$\big|f'(0)\big|^2 \le 4\,\big\|f\big\|_\infty \cdot \big(\|f''\|_\infty + \|f\|_\infty\big),$$

where $\|f\|_\infty = \sup_{t\in[-1,1]} |f(x)|$ denotes the sup-norm.

6.

(a) State, without proof, Tonelli's and Fubini's Theorems.

(b) Prove that if $f \in L^1[0, 1]$ and $a > 0$, then the integral

$$F_a(x) \;=\; \int_0^x (x-t)^{a-1} f(t)dt$$

exists for almost every x in $[0, 1]$ and $F_a \in L^1[0, 1]$.