RETURN THIS COVER SHEET WITH YOUR EXAM AND SOLUTIONS!

Analysis
Ph.D. Preliminary Exam

January 7, 2009

INSTRUCTIONS:

1. Answer each question on a separate page. Turn in a page for each problem even if you cannot do the problem.

2. Label each answer sheet with the problem number.

3. Put your number, not your name, in the upper right hand corner of each page. If you have not received a number, please choose one (1234 for instance) and notify the graduate secretary as to which number you have chosen.

4. Each problem is worth the same number of points. There are six problems.

Analysis Prelim **Jan 2009**

1)

a. Does there exist a real-valued function of a real variable which is continuous at every rational point and discontinuous at every irrational point?

b. Does there exist a real-valued function of a real variable which is discontinuous at every rational point and continuous at every irrational point?

For both parts (a) and (b) if you answer "yes" please provide an example. If you answer "no" please discuss why such an example cannot exist.

2) Let $f(x)$ be a nonnegative, continuous function on $[0,\infty)$ such that $\int_0^\infty f(x)\,dx < \infty$. Show that

$$\lim_{n\to\infty} \frac{1}{t}\int_0^t f(x)\,dx = 0.$$

3) Let $\phi_n, n = 1, 2, \ldots$ be a complete orthonormal system in the space $L^2(-\infty,\infty)$.

a. Show that, for every Borel set $B \subset (-\infty,\infty)$ of strictly positive (finite) Lebesgue measure, one has

$$1 \le \int\limits_B \sum_1^\infty |\phi_n(x)|^2 dx.$$

b. Show that

$$\sum_1^\infty |\phi_n(x)|^2 = \infty \quad \text{a.e.}$$

4) Let f be an integrable function on $(-\infty,\infty)$. Show that

$$\lim_{n\to\infty} \int_{-\infty}^\infty f(x)\sin^2(nx)\ dx = \frac{1}{2}\int_{-\infty}^\infty f(x)\ dx$$

5) Let g be a monotone increasing absolutely continuous function on $[a,b]$ with $g(a) = c,\ g(b) = d$.

a. Show that for any open set $\mathcal{O} \subset [c, d]$

$$m(\mathcal{O}) = \int_{g^{-1}(\mathcal{O})} g'(x)\, dx$$

b. Let $H = \{x : g'(x) \neq 0\}$. If E is a subset of $[c, d]$ with $m(E) = 0$, then $g^{-1}(E) \cap H$ has measure zero.

c. If E is a measurable subset of $[c, d]$ then $F = g^{-1}(E) \cap H$ is measurable and

$$m(E) = \int_F g' = \int_a^b \chi_E(g(x)) \cdot g'(x)\, dx$$

d. Let f be a non-negative measurable function on $[c, d]$, then $(f \circ g)g'$ is measurable on $[a, b]$ and

$$\int_c^d f(y)\, dy = \int_a^b f(g(x))g'(x)\, dx$$

6) Let $f : \mathbb{R} \to \mathbb{R}$ be a function that is continuous at at least one point, such that for all $x, y \in \mathbb{R}$ we have $f(x + y) = f(x) + f(y)$. Show that $f(x) = A \cdot x$ for some real number A. [Hint: Show first that f is continuous everywhere.]