RETURN THIS COVER SHEET WITH YOUR EXAM AND SOLUTIONS!

Analysis

Ph.D.
Preliminary Exam

August, 2018

INSTRUCTIONS:

1. Answer each of the six questions on a separate page. Turn in a page for each problem even if you cannot do the problem.

2. Label each answer sheet with the problem number.

3. Put your number, not your name, in the upper right hand corner of each page. If you have not received a number, please choose one (1234 for instance) and notify the graduate secretary as to which number you have chosen.

1. Let $(X, \mathcal{B}, \mu)$ be a measure space. Let $f : X \to [0, \infty)$ be a μ–measurable function. For each n in $\mathbb{Z}$, set $E_n = \{x \in X : 2^{n-1} < f(x) \leq 2^n\}$. Prove that f is μ-integrable over X if and only if

$$\sum_{n=-\infty}^{+\infty} 2^n \mu(E_n) < \infty.$$

2. Let $u : [0, 1] \to \mathbb{R}$ be absolutely continuous, satisfy $u(0) = 0$, and

$$\int_0^1 |u'(x)|^2 dx < \infty.$$

Prove that

$$\lim_{x \to 0+} \frac{u(x)}{x^{1/2}}$$

exists and determine the value of the limit.

3. Fix $p \in [1, \infty)$, and let $f_n,\ f$ in $L^p[0, 1]$ be such that $\|f_n - f\|_p \to 0$ as $n \to \infty$. Suppose that $\{g_n\}$ is a sequence of Lebesgue measurable functions defined on $[0, 1]$ converging pointwise to the function g on $[0, 1]$, and that M is a fixed positive constant such that $|g_n(x)| \leq M$, $\forall n \in \mathbb{N}$, $\forall\, x \in [0, 1]$. Prove that

$$\lim_{n \to \infty} \|g_n f_n - g f\|_p = 0.$$

4. Show that the space of all continuous functions on the interval $[0, 1]$ with the sup norm $\|f\| = \max_{x \in [0,1]} |f(x)|$ is not a Hilbert space.

5. Let $f : \mathbb{T} \to \mathbb{C}$ be a C^1 function, where $\mathbb{T}$ denotes the one-dimensional torus. If $\int_{-\pi}^{\pi} f(x) dx = 0$, show that

$$\int_{-\pi}^{\pi} |f(x)|^2 dx \leq \int_{-\pi}^{\pi} |f'(x)|^2 dx.$$

6. Let $f : \mathbb{R} \to [0, \infty)$ be Lebesgue integrable. Let m be Lebesgue measure on the real line and $m \times m$ be the product measure on $\mathbb{R}^2$. Prove that $\{(x, y) \in \mathbb{R}^2 : \ 0 \leq y \leq f(x)\}$ is a $m \times m$ measurable subset of $\mathbb{R}^2$ and that

$$m \times m(\{(x, y) \in \mathbb{R}^2 : \ 0 \leq y \leq f(x)\}) = \int_{\mathbb{R}} f(x)\, dx.$$