**Problem 1.** *$f$ is a non-negative real-valued function that belongs to the Lebesgue class $L^n(0,1)$ for every positive integer $n$. Prove that either*

$$\int_0^1 f(x)^n dx \to \infty \quad \text{as } n \to \infty,$$

*or*

$$\lim_{n\to\infty} \int_0^1 f(x)^n dx$$

*exists finitely.*

**Problem 2.** *Prove, from first principles (without appealing to a standard theorem) that with the standard topology on the reals:*

a) *Every open set is the union of a sequence of closed sets.*

b) *Every closed set is the intersection of a sequence of open sets.*

**Problem 3.** *Let $(H, (\cdot,\cdot))$ be a Hilbert space. Recall $u_n \to u$ weakly in $H$ if $(u_n, v) \to (u, v)$ for all $v \in H$. Show if $u_n \to u$ weakly, then $u_n \to u$ (strongly) in $H$ if and only if $\|u_n\| \to \|u\|$.*

**Problem 4.** *Let $C$ and $\alpha$ be two* fixed *positive real numbers. Define $K$ to be the set of all real-valued functions on $[0,1]$ satisfying*

$$|f(x) - f(y)| \leq C|x-y|^\alpha, \quad x, y \in [0,1].$$

*Is $K$ equicontinuous?*

*Then, with respect to the $L^\infty$ norm, is $K$*

- *Closed?*
- *Bounded?*
- *Compact?*

**Problem 5.** *The real-valued function $f$ belongs to the Lebesgue class $L^2(0,1)$ and satisfies*

$$\int_0^1 f(x)dx = 0.$$

*Prove that*

$$\left| \int_0^1 x f(x) dx \right| \leq \frac{1}{2\sqrt{3}} \left\{ \int_0^1 \left| f(x)^2 \right| dx \right\}^{\frac{1}{2}}$$

*Hint: Consider $\int_0^1 |f(x) - \alpha - \beta x|^2 \, dx$, $\alpha$, $\beta$ real.*

**Problem 6.** *Suppose $K(x,y)$ is measurable on $\mathbb{R}^n \times \mathbb{R}^n$ and there is some $C > 0$ such that*

$$\int |K(x,y)| \, dx \leq C, \quad \text{for a.e.} \quad y \in R^n,$$
$$\int |K(x,y)| \, dy \leq C, \quad \text{for a.e.} \quad x \in R^n.$$

*Let*

$$F(x) = \int K(x,y) f(y) dy,$$

*where $f \in L^p(\mathbb{R}^n), 1 \leq p \leq \infty$. Prove $F \in L^p(\mathbb{R}^n)$.*
*Hint: Use Hölder.*