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Analysis

Ph.D.
Preliminary Exam

August, 2016

INSTRUCTIONS:

1. Answer each of the six questions on a separate page. Turn in a page for each problem even if you cannot do the problem.

2. Label each answer sheet with the problem number.

3. Put your number, not your name, in the upper right hand corner of each page. If you have not received a number, please choose one (1234 for instance) and notify the graduate secretary as to which number you have chosen.

1. Let m be Lebesgue measure on $\mathbb{R}$, and let $E \subset \mathbb{R}$ have finite Lebesgue measure. If $E_r = \{x \in E : |x| > r\}$, prove that $m(E_r) \to 0$ as $r \to \infty$.

2. Let $f_n : [0,1] \to \mathbb{R}$ be a sequence of measurable functions. Suppose that

   (i) $\int_0^1 |f_n|^2 \leq 1$ for $n = 1, 2, \ldots$, and

   (ii) $f_n \to 0$ almost everywhere.

   Show that
$$\lim_{n\to\infty} \int_0^1 f_n = 0.$$

3. Let f and g be real-valued integrable functions on a measure space $(X, \mathcal{B}, \mu)$, and define
$$F_t = \{x \in X : f(x) > t\}, \quad G_t = \{x \in X : g(x) > t\}.$$
   Prove that
$$\int |f - g| d\mu = \int_{-\infty}^{\infty} \mu\left((F_t \setminus G_t) \cup (G_t \setminus F_t)\right) dt.$$
   Hint: Rewrite the right-hand side as a double integral.

4. Let $f \in L^1(\mathbb{R})$ be a function satisfying $\int_{\mathbb{R}} |f(x)| dx = 1$.

   (a) Prove that
$$\lim_{|t|\to\infty} \int_{\mathbb{R}} f(x) \cos(tx) dx = 0.$$
   Justify your reasoning.

   (b) Compute
$$\lim_{t\to+\infty} \int_{\mathbb{R}} |f(x) \sin^2(tx)| dx.$$
   Justify your reasoning.

5. (a) Let $f : [0,1] \to \mathbb{R} \cup \{\pm\infty\}$ be in $L^s([0,1])$, where $s \in (1, \infty)$. Suppose that $r \in [1, \infty)$ and $r < s$. Prove that $f \in L^r([0,1])$.

   (b) Prove that $L^6(\mathbb{R}) \cap L^3(\mathbb{R}) \subset L^4(\mathbb{R})$, and moreover show that this containment is proper. Explain your reasoning.

6. Let $C([0,1])$ be the Banach space of all complex-valued continuous functions on $[0,1]$ with norm

$$\|f\| \;=\; \sup_{x\in[0,1]} |f(x)|.$$

   (a) If we define $\mathbf{B}$ by

   $$\mathbf{B} \;=\; \{f \in C([0,1]) :\; \|f\| \leq 1\},$$

   show that $\mathbf{B}$ is a closed subset of $C([0,1])$ that is not compact.

   (b) Let $H : [0,1] \times [0,1] \to \mathbb{C}$ be a continuous function, and for $f \in C([0,1])$ define

   $$S(f)(x) \;=\; \int_0^1 H(x,y)f(y)dy.$$

   Prove that if $f \in C([0,1])$ then $S(f) \in C([0,1])$, and also prove that the closure of $\{S(f) : f \in \mathbf{B}\}$ is compact in $C([0,1])$.

Solutions:

1. Let (r_n) be an increasing sequence of positive real numbers tending to infinity. It suffices to show
$$\lim_{n\to\infty} m(E_{r_n}) = 0.$$
(Observe that the limit on the left-hand side always exists since $m(E_{r_n})$ is a decreasing sequence.) Let $F_n = E \setminus E_{r_n}$. Then $F_1 \subseteq F_2 \subseteq \cdots$ and $m(F_n) = m(E) - m(E_{r_n})$. Thus, by continuity from below,
$$\lim_{n\to\infty} m(F_n) = m(E)$$
since $\cap_n E_{r_n} = \emptyset$. On the other hand,
$$\lim_{n\to\infty} m(F_n) = m(E) - \lim_{n\to\infty} m(E_{r_n})$$
yielding the desired conclusion. $\square$

2. Let $M > 0$. Then
$$\int_0^1 f_n = \int_0^1 f_n\chi_{\{|f_n|\le M\}} + \int_0^1 f_n\chi_{\{|f_n|>M\}},$$
where $\chi_{\{|f_n|\le M\}}$ and $\chi_{\{|f_n|>M\}}$ are indicator functions. Notice that the sequence of functions $(f_n\chi_{\{|f_n|\le M\}})$ is uniformly bounded. So, by the dominated convergence theorem,
$$\lim_{n\to\infty} \int_0^1 f_n\chi_{\{|f_n|\le M\}} = 0.$$
For the second integral,
$$\int_0^1 |f_n|\chi_{\{|f_n|>M\}} \le \frac{1}{M}\int_0^1 |f_n|^2 \le \frac{1}{M},$$
which can be made arbitrarily small by taking M sufficiently large. $\square$

3. Letting χ_t be the indicator function of the set $(F_t \setminus G_t) \cup (G_t \setminus F_t)$ allows us to write
$$\int_{-\infty}^{\infty} \mu\left((F_t \setminus G_t) \cup (G_t \setminus F_t)\right) dt = \int_{-\infty}^{\infty}\int_X \chi_t(x)d\mu(x)dt.$$
Applying Fubini's theorem gives the desired conclusion. $\square$

4. (a) This is just a special case of the Riemann–Lebesgue Lemma, but we include the proof. We consider the case where we have $g = \chi_{[a,b]} \in L^1(\mathbb{R})$, for a finite subinterval $[a,b] \subset \mathbb{R}$. Then

$$\lim_{|t|\to\infty} \int_{\mathbb{R}} g(x) \cos txdx \;=\; \lim_{|t|\to\infty} \int_a^b \cos txdx$$

$$\lim_{|t|\to\infty} [\frac{\sin tx}{t}]_{x=a}^{x=b} \;=\; \lim_{|t|\to\infty} [\frac{\sin tb - \sin ta}{t}].$$

We now note that

$$||[\frac{\sin tb - \sin ta}{t}]|| \leq \frac{2}{|t|}$$

so that

$$0 \leq \limsup_{|t|\to\infty} ||[\frac{\sin tb - \sin ta}{t}]|| \leq \lim_{|t|\to\infty} \frac{2}{|t|} = 0.$$

Therefore

$$\lim_{|t|\to\infty} \int_{\mathbb{R}} g(x) \cos txdx \;=\; 0.$$

Now let $[a_i, b_i] \subset \mathbb{R}$, $1 \leq i \leq n$, be a collection of finite intervals in $\mathbb{R}$, and let $\{\alpha_i\}I = 1^n \subset \mathbb{C}$. Let $g_i = \chi_{[a_i,b_i]} \in L^1(\mathbb{R})$. By properties of the integral and of limits we get:

$$\lim_{|t|\to\infty} \int_{\mathbb{R}} [\sum_{i=1}^n \alpha_i g_i(x)] \cos txdx$$

$$= \lim_{|t|\to\infty} \sum_{i=1}^n \alpha_i \int_{\mathbb{R}} g_i(x)dx$$

$$= \sum_{i=1}^n \alpha_i [\lim_{|t|\to\infty} \int_{\mathbb{R}} g_i(x)dx] \;=\; \sum_{i=1}^n \alpha_i \cdot 0 \;=\; 0.$$

We have thus shown that if $\psi \in L^1(\mathbb{R})$ is a step function,

$$\lim_{|t|\to\infty} \int_{\mathbb{R}} \psi(x) \cos txdx = 0.$$

But step functions are dense in $L^1(\mathbb{R})$ in the L^1 norm, so that given $f \in L^1(\mathbb{R})$ with $\|f\|_1 = 1$, there is a sequence of step functions $\{psi_k\}_{k=1}^\infty$ with

$$\lim_{k\to\infty} \int_{\mathbb{R}} |f(x) - \psi_k(x)|dx = 0.$$

Given $\varepsilon > 0$, find $K > 0$ such that for all $k \geq K$,

$$\int_{\mathbb{R}} |f(x) - \psi_k(x)| dx < \frac{\varepsilon}{2}.$$

In particular,

$$\int_{\mathbb{R}} |f(x) - \psi_K(x)| dx < \frac{\varepsilon}{2}.$$

Note ψ_K is a step function, so there exists $N > 0$ such that if $|t| > N$,

$$|\int_{\mathbb{R}} \psi_K(x) \cos txdx| < \frac{\varepsilon}{2}.$$

So, for $|t| > N$,

$$\begin{aligned}
|\int_{\mathbb{R}} f(x) \cos txdx; =& \ |\int_{\mathbb{R}} [f(x) - \psi_K(x) + \psi_K(x)] \cos txdx| \\
=& \ |\int_{\mathbb{R}} [f(x) - \psi_K(x)] \cos txdx + \int_{\mathbb{R}} \psi_K(x) \cos txdx| \\
\leq& \ |\int_{\mathbb{R}} [f(x) - \psi_K(x)] \cos txdx| + |\int_{\mathbb{R}} \psi_K(x) \cos txdx| \\
\leq& \ \int_{\mathbb{R}} |(f(x) - \psi_K(x)) \cos tx| dx + |\int_{\mathbb{R}} \psi_K(x) \cos txdx| \\
\leq& \ \int_{\mathbb{R}} |(f(x) - \psi_K(x))| dx + +|\int_{\mathbb{R}} \psi_K(x) \cos txdx| \\
<& \ \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

We thus have shown that

$$\lim_{|t| \to \infty} \int_{\mathbb{R}} f(x) \cos txdx = 0,$$

as desired. $\square$

(b) We use the following double-angle formula in trigonometry:

$$\sin^2 tx \ = \ \frac{1 - \cos [2tx]}{2}.$$

Therefore since $\sin^{tx} | \geq 0$,

$$\lim_{t \to +\infty} \int_{\mathbb{R}} |f(x) \sin^2 tx| dx \ = \ \lim_{t \to +\infty} \int_{\mathbb{R}} |f(x)| \sin^2 txdx$$

$$= \lim_{t\to+\infty}\int_{\mathbb{R}} |f(x)|\frac{1-\cos[2tx]}{2}dx \;=\; \int_{\mathbb{R}} |f(x)|\frac{1}{2}dx - \lim_{t\to+\infty}\int_{\mathbb{R}} |f(x)|\frac{\cos[2tx]}{2}dx$$

$$\frac{1}{2} - \frac{1}{2}\cdot \lim_{t\to+\infty}\int_{\mathbb{R}} |f(x)|\cos[2tx]dx$$

(the first "$\frac{1}{2}$" occurring since we are told that $\|f\|_1 = 1$)

$$= \frac{1}{2} - 0$$

(by part (a), since $|f| \in L^1(\mathbb{R})$). Therefore, we have proved that

$$\lim_{t\to+\infty}\int_{\mathbb{R}} |f(x)\sin^2 tx|dx \;=\; \frac{1}{2}.$$

$\square$

5. (a) Since $f \in L^s([0,1])$, we have

$$\int_{[0,1]} |f(x)|^s dx \;=\; \int_{[0,1]} (|f(x)|^{\frac{s}{r}})^r dx < \infty.$$

It follows that $g(x) = |f(x)|^r$ is an element of $L^{\frac{s}{r}}([0,1])$. We now apply Hölder's inequality to the product $|f|^r \cdot 1 = |f|^r$ with respect to the conjugate exponents $\frac{s}{r}$ and $\frac{1}{1-\frac{r}{s}} = \frac{s}{s-r} = p'$ to obtain

$$\int_{[0,1]} |f(x)|^r dx \;=\; \int_{[0,1]} |f(x)|^r \cdot 1 \; dx$$

$$\leq \; [\int_{[0,1]} (|f(x)|^r)^{\frac{s}{r}} dx]^{\frac{r}{s}} \cdot [\int_{[0,1]} (1)^{\frac{s}{s-r}} dx]^{\frac{s-r}{s}}$$

$$= \; [\int_{[0,1]} |f(x)|^s dx]^{\frac{r}{s}} \cdot 1 \; < \; \infty.$$

Thus $f \in L^r([0,1])$. $\square$

(b) We first show that

$$\|f\|_4 \;\leq\; \max\{\|f\|_3, \|f\|_6\}.$$

We write 4 as a convex combination of 3 and 6 :

$$4 \;=\; (1-\frac{1}{3})\cdot 3 + \frac{1}{3}\cdot 6.$$

So

$$|f(x)|^4 \;=\; |f(x)|^{\frac{2}{3}\cdot 3 + \frac{1}{3}\cdot 6} \;=\; |f(x)|^2 \cdot |f(x)|^2.$$

We now apply Hölder's inequality with $p = \frac{3}{2},\ q = 3$ to obtain that

$$\int_{\mathbb{R}} |f(x)|^4 dx \ = \ \int_{\mathbb{R}} |f(x)|^2 \cdot |f(x)|^2 dx$$
$$\leq \ [\int_{\mathbb{R}} [|f(x)|^2]^{\frac{3}{2}} dx]^{\frac{2}{3}} \cdot [\int_{\mathbb{R}} [|f(x)|^2]^3 dx]^{\frac{1}{3}}$$
$$= \ [\int_{\mathbb{R}} |f(x)|^3 dx]^{\frac{2}{3}} \cdot [\int_{\mathbb{R}} |f(x)|^6 dx]^{\frac{1}{3}}$$
$$= \ (\|f\|_3)^2 \cdot (\|f\|_6)^2.$$

It follows that

$$\|f\|_4 \ = \ [\int_{\mathbb{R}} |f(x)|^4 dx]^{\frac{1}{2}} \ \leq \ (\|f\|_3)^{\frac{1}{2}} \cdot (\|f\|_6)^{\frac{1}{2}}.$$

Let $M \ = \ \max\{\|f\|_3, \|f\|_6\}$. We then obtain

$$\|f\|_4 \ \leq \ (\|f\|_3)^{\frac{1}{2}} \cdot (\|f\|_6)^{\frac{1}{2}} \ \leq \ M^{\frac{1}{2}} \cdot M^{\frac{1}{2}} \ = \ M.$$

Hence $\|f\|_4 \ \leq \ \max\{\|f\|_3, \|f\|_6\}$, as we desired to show. Now suppose $f \in L^6(\mathbb{R}) \cap L^3(\mathbb{R})$. Then $\|f\|_6 < \infty$ and $\|f\|_3 < \infty$. Hence $M \ = \ \max\{\|f\|_3, \|f\|_6\} < \infty$. Since $\|f\|_4 \leq M$, we get the desired result, that $\|f\|_4 < \infty$, so that $f \in L^4(\mathbb{R})$. Finally, to show the containment is proper, consider the function

$$f(x) = \chi_{(0,1)}(x) \frac{1}{x^{2/9}}$$

We note that

$$|f(x)|^4 = \chi_{(0,1)}(x) [\frac{1}{x^{2/9}}]^4 \ = \ \chi_{(0,1)}(x) \frac{1}{x^{8/9}}$$

so that

$$\int_{\mathbb{R}} |f(x)|^4 dx \ = \ \int_{[0,1]} \frac{1}{x^{8/9}} < \infty.$$

Therefore $f \in L^4(\mathbb{R})$. On the other hand,

$$|f(x)|^6 = \chi_{(0,1)}(x) \frac{1}{x^{12/9}}$$

so that

$$\int_{\mathbb{R}} |f(x)|^6 dx \ = \ \int_{[0,1]} \frac{1}{x^{12/9}} = \infty.$$

Therefore $f \notin L^6(\mathbb{R})$, so that $f \notin L^6(\mathbb{R}) \cap L^3(\mathbb{R})$. (The exponent $2/9$ was chosen since $1/6 < 2/9 < 1/4$.)

$\square$

6. (a) We first show that $\mathbf{B}$ is closed. Recall the sup norm on $C([0,1])$ is a norm and the norm is continuous since we have

$$|\|f\| - \|g\|| \ \leq \ \|f-g\|, \ \forall f, g \in C([0,1]),$$

by standard properties of norm. Therefore if $\{g_n\}$ is a Cauchy sequence in $\mathbf{B}$. by completeness of $C([0,1])$ this sequence will converge in norm to some $g \in C([0,1])$, Since the norm is continuous,

$$\lim_{n\to\infty} \|g_n\| \ = \ \|g\|.$$

But for all $n \in \mathbb{N}, \ g_n \in \mathbf{B}$, so that $\|g_n\| \leq 1, \ \forall n \in \mathbb{N}$. Therefore $\|g\| \leq 1$, so that $g \in \mathbf{B}$, and $\mathbf{B}$ is closed.

Now consider the sequence of functions in $\mathbf{B}$ given by $\{f_n(x) = x^n\}_{n=1}^{\infty}$. We calculate the pointwise limit of the f_n :

$$\lim_{n\to\infty} f_n(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1, \\ 1 & \text{if } x = 1. \end{cases}$$

We see that the pointwise limit function is discontinuous at $x = 1$. Therefore, the sequence $\{f_n\}$ does not have a subsequence that converges uniformly, so that $\{f_n\}$ is a sequence in $\mathbf{B}$ that does not have any convergent subsequence in norm, so $\mathbf{B}$ cannot be compact. $\square$

(b) Fix $f \in \mathbf{B}$. We first show that $S(f)$ is continuous. Let $\varepsilon > 0$ be fixed. We note that since H is continuous on the compact set $[0,1] \times [0,1]$, it is uniformly continuous there, so that there exists $\delta > 0$ such that whenever $x_1, \ x_2, \ y_1, \ y_2 \in [0,1]$ and $|x_1 - x_2| < \delta$ and $|y_1 - y_2| < \delta, \ |H(x_1, y_1) - H(x_2, y_2)| < \frac{\epsilon}{2}$. So if $x_1, \ x_2 \in [0,1]$ and $|x_1 - x_2| < \delta$,

$$|S(f)(x_1) - S(f)(x_2)| \ = \ |\int_0^1 H(x_1, y) f(y) dy - \int_0^1 H(x_2, y) f(y) dy|$$

$$= \ |\int_0^1 (H(x_1, y) - H(x_2, y)) f(y) dy| \ \leq \ \int_0^1 |H(x_1, y) - H(x_2, y)||f(y)| dy$$

$$\leq \int_0^1 \frac{\varepsilon}{2} \cdot 1 dy = \frac{\varepsilon}{2} < \varepsilon.$$

(We recall that $f \in \mathbf{B}$ so that $\sup_{x\in[0,1]} |f(x)| = \|f\| \leq 1$.) Therefore $S(f)$ is uniformly continuous on $[0,1]$ so that $S(f) \in C([0,1])$. The above argument also shows that the set $\{S(f) : f \in \mathbf{B}\}$ is equicontinuous, because the value δ chosen above depends only on ε and H and is independent of $f \in \mathbf{B}$. Since H

is continuous on the compact set $[0,1] \times [0,1]$ it is bounded on that set, and therefore there exists $M > 0$ such that

$$|H(x,y)| \leq M \;\forall\; (x,y) \;\in [0,1] \times [0,1].$$

It follows that for $f \in \mathbf{B}$, and $x \in [0,1]$,

$$|S(f)(x)| \;=\; |\int_0^1 H(x,y)f(y)dy| \;\leq\; \int_0^1 |H(x,y)||f(y)|dy \;\leq\; \int_0^1 M \cdot 1dy \;= M.$$

Thus $\{S(f) : f \in \mathbf{B}\}$ is equicontinuous and pointwise (in fact uniformly) bounded on $[0,1]$, so that by the Arzela–Ascoli Theorem, the closure of $\{S(f) : f \in \mathbf{B}\}$ is compact in $C([0,1])$. $\square$