RETURN THIS COVER SHEET WITH YOUR EXAM AND SOLUTIONS!

Analysis

Ph.D. Preliminary Exam
Department of Mathematics
University of Colorado Boulder

August, 2015

INSTRUCTIONS:

1. Answer each of the six questions on a separate page. Turn in a page for each problem even if you cannot do the problem.

2. Label each answer sheet with the problem number.

3. Put your number, not your name, in the upper right hand corner of each page. If you have not received a number, please choose one (1234 for instance) and notify the graduate secretary as to which number you have chosen.

Analysis Prelim **August, 2015**

Q.1 Suppose that $f : [0,1] \to \mathbb{R}$ is continuous. Prove that

$$\lim_{n\to\infty} \int_0^1 f(x^n)dx$$

exists and evaluate the limit. Does the limit always exist if f is only assumed to be Lebesgue integrable?

Q.2 Assume that a Lebesgue measurable set E is contained in the interval $[a,b]$ for some $0 < a < b < \infty$. Let $\delta > 1$. If the sets E and δE (the elements of E each multiplied by δ) are disjoint, prove that the measure of E is at most $\frac{b}{2}\log(b\delta/a)$.

Q.3 (i) Find a sequence of continuous functions on $[0,1]$ converging pointwise but not uniformly.

(ii) Prove that the space $C([0,1])$ of continuous functions on $[0,1]$ is not complete in the L^1 metric $d(f,g) = \int_0^1 |f(x) - g(x)|dx$.

Q.4 Let $\{\phi_n\}$ be a sequence of continuous real-valued functions defined on a compact metric space X. For each $x \in X$, suppose that the sequence of values $\{\phi_n(x)\}$ is non-decreasing and bounded above. Define

$$\phi(x) = \lim_{n\to\infty} \phi_n(x).$$

If ϕ is continuous, prove that the sequence $\{\phi_n\}$ converges uniformly to ϕ.

Q.5 Let M be a bounded subset of $C([a,b])$, the set of continuous functions on $[a,b]$ equipped with the sup norm. Set

$$A = \left\{ F : [a,b] \to \mathbb{R} : F(x) = \int_a^x f(t)dt \text{ for some } f \in M \right\}.$$

Show that the closure of A is a compact subset of $C([a,b])$.

Q.6 Let f be a Lebesgue measurable real-valued function on the interval $(0,1)$. For $n = 1, 2, \ldots$, assume that the integrals

$$\int_0^1 x\,(f(x))^n\,dx$$

exist and have the same non-zero value. Prove that $f(x) = 1$ on a set of positive measure and is otherwise almost everywhere zero. [**Hint:** First show that f is essentially bounded.]