RETURN THIS COVER SHEET WITH YOUR EXAM AND SOLUTIONS!

Analysis

Ph.D. Preliminary Exam
Department of Mathematics
University of Colorado Boulder

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INSTRUCTIONS:

1. Answer each of the six questions on a separate page. Turn in a page for each problem even if you cannot do the problem.

2. Label each answer sheet with the problem number.

3. Put your number, not your name, in the upper right hand corner of each page. If you have not received a number, please choose one (1234 for instance) and notify the graduate secretary as to which number you have chosen.

Q.1 Prove the following statement or provide a counterexample to the following statement: There exists an open subset, E, of the closed unit interval on the real line, $[0,1]$, with the following two properties:

(a) Lebesgue measure of $\{E \cap (a,b)\} > 0$ for all non-empty open subintervals, (a,b), of $[0,1]$ with $0 < a < b < 1$;

(b) Lebesgue measure of $E < 1$.

Q.2 Let $f \in L^p(-\infty,\infty)$ where $1 \le p \le \infty$. Show that the function

$$F(t) = \int_0^t f(s)\, ds$$

is well defined and continuous.

Q.3 Let $H_k(t), k = 0,1,2,\ldots$ be a sequence of functions on $[0,1]$ defined as follows: $H_0(t) \equiv 1$ and, if $2^n \le k < 2^{n+1}$ where n is a nonnegative integer, then

$$H_k(t) = \begin{cases} 2^{n/2} & \text{if } \frac{k-2^n}{2^n} \le t < \frac{k-2^n+0.5}{2^n} \\ -2^{n/2} & \text{if } \frac{k-2^n+0.5}{2^n} \le t < \frac{k-2^n+1}{2^n} \\ 0 & \text{otherwise} \end{cases}$$

Show that, for every function f in the Hilbert space $L^2[0,1]$,

$$\lim_{k\to\infty} \int_0^1 f(t) H_k(t)\, dt = 0$$

Q.4 Consider the expression

$$\int_0^\infty \frac{\sin x}{x^\alpha} dx.$$

Does there exist an $\alpha > 0$ such that the given integral expression exists as an improper Riemann integral but does not exist as a Lebesgue integral? Prove your answer.

Q.5 Let A_k be a sequence of measurable subsets of $[0,1]$ such that, for every finite set of indices $i_1 < i_2 < \cdots < i_k$,

$$m(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}) = m(A_{i_1}) m(A_{i_2}) \ldots m(A_{i_k})$$

where m stands for the Lebesgue measure.

(a) Show that the sequence $B_k = [0,1] \setminus A_k$ has the same property. (*Hint:* Show that, if the property holds for the sequence A_k, then it still holds if exactly one of the sets A_k is replaced by the corresponding B_k).

(b) Suppose in addition that the series $\sum m(A_k)$ diverges. Show that

$$m\left(\bigcup_{k=1}^{\infty} A_k\right) = 1$$

Q.6 Let μ_s be Lebesgue measure on $S = [0, 1]$; let μ_t be the counting measure on S, i.e., $\mu_t(B) =$ the number of elements of B, for any finite $B \subset S$. Let D be the diagonal, $D = \{(s, t) : s = t\}$ in $S \times S$, and f the characteristic function of D.

(a) Show that, for any s,

$$\int_S f(s, t)\, d\mu_t = 1,$$

(b) Show that, for any t,

$$\int_S f(s, t)\, d\mu_s = 0,$$

(c) Show that

$$\int_{S \times S} f(s, t)\, d\mu,$$

where $\mu = \mu_s \otimes \mu_t$, does not exist (as a finite number).