Department of Mathematics, University of Colorado Boulder

Analysis Preliminary Examination

August 2013

The six problems have equal points. Please do all of them.

1. Prove the following "modified squeeze law:"

   Suppose we have real numbers a_n, $b_{m,n}$, and c_m such that
   $$0 \leq a_n \leq b_{m,n} + c_m$$
   for all m, n sufficiently large (say, greater than some fixed integer K). If
   $$\lim_{m\to\infty} c_m = 0$$
   and, for any fixed m,
   $$\lim_{n\to\infty} b_{m,n} = 0,$$
   then
   $$\lim_{n\to\infty} a_n = 0.$$

2. Prove or disprove the following:

   Let $f, g\colon \mathbb{R} \to \mathbb{R}$, and suppose the composite function $f \circ g$ is continuous everywhere. If $\lim_{u\to b} f(u) = c$ and $\lim_{x\to a} g(x) = b$ ($b, c \in \mathbb{R}$), then $\lim_{x\to a} f(g(x)) = c$.

3. The *convolution*, denoted $f * g$, of two functions $f, g\colon \mathbb{R} \to \mathbb{R}$ is defined by
   $$f * g(x) = \int_{\mathbb{R}} f(x-y)g(y)\,dy,$$
   for any $x \in \mathbb{R}$ such that the integral on the right exists (in the Lebesgue sense).

   (a) Show that, if $f, g \in L^2(\mathbb{R})$, then $f * g(x)$ exists for all x, that $f * g$ is bounded on $\mathbb{R}$, and that
   $$\sup_{x\in\mathbb{R}} |f * g(x)| \leq ||f||_2 \cdot ||g||_2$$
   (here, $||\cdot||_2$ denotes the norm on $L^2(\mathbb{R})$).

   (b) Show that, if $f, g \in L^1(\mathbb{R})$, then $f * g(x)$ exists for almost all x, that $f * g \in L^1(\mathbb{R})$, and that
   $$||f * g||_1 \leq ||f||_1 \cdot ||g||_1$$
   (here, $||\cdot||_1$ denotes the norm on $L^1(\mathbb{R})$).

4. The *Fourier transform*, denoted $\widehat{f}$, of a function $f\colon \mathbb{R} \to \mathbb{R}$ is defned by
   $$\widehat{f}(s) = \int_{\mathbb{R}} f(x)e^{-2\pi isx}\,dx,$$
   for any $s \in \mathbb{R}$ such that the integral on the right exists (in the Lebesgue sense).

(a) Show that, if $f \in L^1(\mathbb{R})$, then $\widehat{f}(s)$ exists for all s, that $\widehat{f}$ is bounded and continuous, and that

$$\sup_{s \in \mathbb{R}} |\widehat{f}(s)| \leq ||f||_1$$

(here, $|| \cdot ||_1$ denotes the norm on $L^1(\mathbb{R})$).

(b) Show that, if $f, g \in L^1(\mathbb{R})$, then

$$\int_{\mathbb{R}} \widehat{f}(u) g(u)\, du = \int_{\mathbb{R}} f(v) \widehat{g}(v)\, dv.$$

5. Give an example of a subset of $\mathbb{R}$ that is not a G_δ set. (Recall that a G_δ set in $\mathbb{R}$ is a countable intersection of open subsets of $\mathbb{R}$.) Can such a set be countable? If so, give an example to show this. If not, explain why not.

6. Prove or disprove the following:

   Let A be a measurable subset of $\mathbb{R}$. Let $I = A \cap [a, b]$, where $[a, b]$ is a compact interval in $\mathbb{R}$, and let $f \colon I \to \mathbb{R}$. Assume f is continuous on I, in the sense that

$$\lim_{n \to \infty} x_n = x \Rightarrow \lim_{n \to \infty} f(x_n) = f(x),$$

   for $x_n, x \in I$. Then f is bounded.