RETURN THIS COVER SHEET WITH YOUR EXAM AND SOLUTIONS!

Analysis

Ph.D. Preliminary Exam

August, 2012

INSTRUCTIONS:

1. Answer each of the six questions on a separate page. Turn in a page for each problem even if you cannot do the problem.

2. Label each answer sheet with the problem number.

3. Put your number, not your name, in the upper right hand corner of each page. If you have not received a number, please choose one (1234 for instance) and notify the graduate secretary as to which number you have chosen.

Analysis Prelim **August, 2012**

1. Let $\{f_n : n \in \mathbb{N}\}$ be a sequence of real-valued Lebesgue measurable functions defined on $[0,1]$. Suppose $\lim_{n\to\infty} f_n(x) = f(x)$ for almost all $x \in [0,1]$.

(a) Is f necessarily Lebesgue measurable? If yes, prove it, and if no, provide a counterexample.

(b) Give a condition on $\{f_n : n \in \mathbb{N}\}$ that guarantees

$$\lim_{n\to\infty} \int_0^1 f_n = \int_0^1 f.$$

Be sure to prove that your condition implies the desired conclusion.

(c) Give an example of a sequence of Lebesgue measurable functions $\{f_n\}$ defined on $[0,1]$ that violates your condition in (b) and such that

$$\lim_{n\to\infty} \int_0^1 f_n \neq \int_0^1 f.$$

2. Let $f \in L^1(\mathbb{R})$, the set of Lebesgue integrable functions over $\mathbb{R}$. Prove that

$$\lim_{x\to 0} \int_{\mathbb{R}} |f(t+x) - f(t)| dt = 0.$$

You may use the fact that the space $C_C(\mathbb{R})$ of continuous functions on $\mathbb{R}$ with compact support is dense in $L^1(\mathbb{R})$, with respect to the norm

$$\|g\|_1 = \int_{\mathbb{R}} |g(t)| dt.$$

3. Let f be a measurable function on $\mathbb{R}$ with $f \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$.

(a) Prove that for all $p \in (1,\infty)$, $f \in L^p(\mathbb{R})$.

(b) Prove that

$$\lim_{p\to\infty} \|f\|_p = \|f\|_\infty.$$

4. A function $f : [a, b] \to \mathbb{R}$ is said to be **Lipschitz** on $[a, b]$ provided there is a constant $M > 0$ such that $|f(x) - f(y)| \leq M|x - y|$ for all $x, y \in [a, b]$.

(a) Prove that if $g : [a, b] \to [c, d]$ is absolutely continuous on $[a, b]$, and $f : [c, d] \to \mathbb{R}$ is Lipschitz on $[c, d]$, then $f \circ g : [a, b] \to \mathbb{R}$ is absolutely continuous on $[a, b]$.

(b) By using part (a) or otherwise, prove that any Lipschitz function f defined on $[a, b]$ is absolutely continuous. Is the converse true, i.e. is an absolutely continuous function $f : [a, b] \to \mathbb{R}$ necessarily Lipschitz? Either prove this is true, or provide a counterexample.

5. Let $f \in L^1[-\pi, \pi]$, and for $n \in \mathbb{Z}$, define $c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)e^{-int}dt$, where $e^{i\theta} = \cos\theta + i \sin\theta$.

(a) Prove that $\lim_{|n|\to\infty} c_n$ exists.

(b) Is the limit in (a) independent of f? If so, prove it. If no, give examples of f_1 and $f_2 \in L^1[-\pi, \pi]$ with different limits arising in (a).

6. Let $f : [0, 1] \to \mathbb{R}$ be continuous with $f(0) = f(1)$. Prove that there exists $x \in [0, \frac{3}{4}]$ with $f(x) = f(x + \frac{1}{4})$.