1. Let m be the Lebesgue measure on the real line, and let B be a Borel subset with $m(B) < \infty$.

(a) Show that, for every $1 \le p < \infty$, there exists a sequence of continuous functions ϕ_n with compact supports such that $\phi_n \to 1_B$ in L^p (show details).

(b) Show that continuous functions with compact support are dense in $L^p(R)$.

2. Let $\rho_y(x) = \frac{1}{\pi}\frac{y}{x^2+y^2}$ where $x \in R$ and $y > 0$. Given a bounded uniformly continuous function f on R, let

$$u_f(x,y) = \int_{-\infty}^{\infty} \rho_y(x-z)f(z)dz$$

Show that

$$|u_f(x,y) - f(x)| \le \omega_f(\delta) + 2\|f\|_\infty(1 - \frac{2}{\pi}\arctan\frac{\delta}{y})$$

where

$$\omega_f(\delta) = \sup_{x,x' \in R, |x-x'|<\delta} \{|f(x) - f(x')|\}$$

and $\|\cdot\|_\infty$ stands for L^∞ norm. In particular, conclude from this that $u_f(x,y) \to f(x)$ uniformly as $y \to 0$ for such f.

3. Let μ be a finite measure on $(0,1)$ such that μ and Lebesgue measure are mutually singular. Show that, for every $\varepsilon \in (0, \mu(0,1))$, there exists a finite collection of disjoint open intervals (x_k, y_k) such that $\sum |y_k - x_k| < \varepsilon$ and $\sum \mu(x_k, y_k) \ge \mu(0,1) - \varepsilon$.

4. Let (R, d_1) be the metric space which is the real line R with usual complete Euclidean metric $d_1(x,y) = |x-y|$ for all $x, y \in R$. If d_2 is another metric on R such that (R, d_2) is a metric space with the same topology as (R, d_1), can we conclude that (R, d_2) is a complete metric space? Justify your answer. [Hint: Consider the function $\phi(x) = \frac{x}{1+|x|}$ and $d_2(x,y) = |\phi(x) - \phi(y)|$.]

5. Let $L^2(-\pi,\pi)$ be the space of (complex-valued) absolutely square integrable functions on $(-\pi,\pi)$ with respect to Lebesgue measure. Denote

$$E_n(x) = e^{inx}, \quad n = 0, \pm 1, \pm 2, \ldots, -\pi \le x \le \pi$$

and

$$F_n(x) = E_{-n}(x) + nE_n(x), \quad n = 1, 2, 3, \ldots, -\pi \le x \le \pi$$

Let X_1 be the smallest closed subspace of $L^2(-\pi,\pi)$ that contains $E_0, E_1, E_2, \ldots$. Let X_2 be the smallest closed subspace of $L^2(-\pi,\pi)$ that contains $F_1, F_2, \ldots$.

(a) Is $X_1 + X_2$, that is the linear span of X_1 and X_2, dense in $L^2(-\pi,\pi)$? Recall that if A and B are linear subspaces then $A + B = \{a + b : a \in A, b \in B\}$.

(b) Is $X_1 + X_2$ closed? [Hint: consider $\sum_{n=1}^{\infty} n^{-1}E_n$]

6. For a function $f \in L^p[0,\infty), 1 < p < \infty$, set

$$F(x) = \frac{1}{x}\int_0^x f(t)\ dt$$

(a) Assuming f is continuous nonnegative vanishing outside of a finite interval, show that

$$\int_0^\infty F^p(t)\ dt = \frac{p}{p-1} \int_0^\infty F^{p-1}(t) f(t)\ dt.$$

[Hint: consider an equation relating F, f and F'.]

(b) For f as in part (a), establish Hardy's inequality:

$$\|F\|_p \leq \frac{p}{p-1} \|f\|_p.$$

[Hint: Try Hölder inequality.]

(c) Extend Hardy's inequality to all $f \in L^p[0, \infty)$.

Solutions

1. (a) For every $\epsilon > 0$, there exists a compact set $K_\epsilon \in B$ such that $m(B \setminus K_\epsilon) < \epsilon$ and therefore $\|1_B - 1_{K_\epsilon}\|_p < \epsilon$. For a compact set K, consider functions $\phi_\delta(x) = \max\{1 - \frac{1}{\delta}\rho(x, K), 0\}$. They are continuous, they vanish outside of a δ-neighborhood U_δ of K and their norm does not exceed $m(U_\delta \setminus K)$. It remains to cook an appropriate sequence.

(b) Using part (a), we can approximate every simple function that vanishes outside of a set of finite measure. Those simple functions are dense in $L^p(R)$

2. Indeed, note that

$$u_f(x,y) - f(x) = \int_{-\infty}^{\infty} \rho_y(x-z)(f(z) - f(x))\ dz$$

Fix x and denote by U_δ the δ-neighborhood of x. We have

$$|u_f(x,y) - f(x)| \le \int_{-\infty}^{\infty} \rho_y(x-z)|f(z) - f(x)|\ dz = \int_{U_\delta} + \int_{U_\delta^c}$$

Now,

$$\int_{U_\delta} \rho_y(x-z)|f(z) - f(x)|\ dz \le \omega_f(\delta) \int_{U_\delta} \rho_y(x-z)\ dz \le \omega_f(\delta)$$

and

$$\int_{U_\delta^c} \rho_y(x-z)|f(z)-f(x)|\ dz \le 2\|f\|_\infty \int_{U_\delta^c} \rho_y(x-z)\ dz \le 2\|f\|_\infty(1 - \frac{2}{\pi}\arctan\frac{\delta}{y})$$

as promised. Choosing, first, δ such that $\omega_f(\delta) < \varepsilon/2$ and then, y that makes the second part less than $\varepsilon/2$, we get a bound $|u_f(x,y) - f(x)| < \varepsilon$ uniformly.

3. Let $\mu[0,1] = M > 0$. There exists a set B of Lebesgue measure zero, such that $\mu(B^c) = 0$. For every $\varepsilon > 0$, there exists an open set $U \supset B$ such that $m(U) < \varepsilon$ where m stands for Lebesgue measure. Next, U is a union of (at most countable) collection of disjoint open intervals. Since $B \subset U$ and $\mu(B^c) = 0$, we have $\mu(U) = M$ and therefore we can find a finite collection of disjoint open intervals (x_k, y_k) such that $\sum_{k=1}^N \mu(x_k, y_k) \ge M - \varepsilon$ as promised.

4. Let $d_2(x,y) = |\phi(x) - \phi(y)|$ where $\phi(x) = \frac{x}{1+|x|}$ for all $x \in R$.

(a) Both metrics define the same topology:

$$\begin{aligned}
\left|\frac{x}{1+|x|} - \frac{y}{1+|y|}\right| &\le \frac{|z(1+|y|) - y(1+|x|)|}{(1+|x|)(1+|y|)} \\
&\le |x - y + x|y| - y|x|| \\
&\le |x - y + x|y| - y|y| + y|y| - y|x|| \\
&\le |x-y| + |y||x-y| + |y|(|y| - |x|)| \\
&\le |x-y|(1 + 2|y|)
\end{aligned}$$

and therefore d_1 convergence implies d_2 convergence. The other way, we have to consider three cases.

Case 1. Suppose $x, y \geq 0$ and $x, y, \leq M$ for some fixed M. Then $d_2(x, y) \geq \frac{|x-y|}{(1+M)^2}$.

Case 2. Suppose $x, y < 0$ and $x, y, \geq -M$ for some fixed M. Then again, $d_2(x, y) \geq \frac{|x-y|}{(1+M)^2}$.

Case 3. Suppose x and y have opposite signs. Then $d_2(x, y)$ is small if both x and y are close to zero, say, $|x|, |y| < 1$ and then again, $d_2(x, y) \geq |x - y|/2$.

Therefore, id $z_n \to z$ in d_1, then the same is true in d_2 and vice versa.

(b) Consider a sequence $z_n = n$. One can show that z_n forms a Cauchy sequence in d_2, but there is no $x \in R$ such that $d_2(z_n, x) \to 0$ (otherwise $d_1(z_n, x) = |n - x| \to 0$ as well). Therefore, it is not complete.

5. $X_1 + X_2$ contains $E_0, E_1, E_2, \ldots$ and $E_{-k} = F_k - kE_k, k = 1, 2, \ldots$ and therefore it is dense in L^2. However, it is not closed. Indeed, suppose

$$\sum_{n=1}^{\infty} n^{-1} E_{-n} = \sum_{j=0}^{\infty} a_j E_j + \sum_{k=1}^{\infty} b_k (E_{-k} + kE_k)$$

where the first series on the right represents an element of X_1 and the second one belongs to X_2. Taking inner product with E_{-k} and then with E_k, we can find $b_k = k^{-1}, k = 1, 2, \ldots$ and $a_0 = 0, a_j = -1, j = 1, 2, \ldots$. But then, the series $\sum_{j=0}^{\infty} a_j E_j = -\sum_{j=1}^{\infty} E_j$ does not converge.

6. (a) This is just Calculus. First of all, note that

$$xF'(x) = f(x) - F(x)$$

Next, integrating by parts, we get

$$\int_0^{\infty} F^p(t)\, dt = -p \int_0^{\infty} F^{p-1}(t) t F'(t)\, dt = -p \int_0^{\infty} F^{p-1}(t) f(t)\, dt + p \int_0^{\infty} F^p(t)\, dt$$

which implies the statement

(b) Applying the Hölder inequality and taking into account that F is also non-negative, we get

$$\int_0^{\infty} F^{p-1}(t) f(t)\, dt \leq \|F^{p-1}\|_q \|f\|_p$$

where $1/q + 1/p = 1$. Note that $q = \frac{p}{p-1}$ and therefore

$$\|F^{p-1}\|_q = \left(\int (F(t))^{(p-1)q}\, dt \right)^{1/q} = \left(\int (F(t))^p\, dt \right)^{(p-1)/p} = \|F\|_p^{p-1}$$

Therefore part (a) implies

$$\|F\|_p^p \leq \frac{p}{p-1} \|F\|_p^{p-1} \|f\|_p$$

which is equivalent to (b).

(c) First, extend this to linear combinations. Next, continuous functions with compact support are dense in $L^p[0, \infty)$.