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Analysis

Ph.D. Preliminary Exam

August 2010

INSTRUCTIONS:

1. Answer each question on a separate page. Turn in a page for each problem even if you cannot do the problem.

2. Label each answer sheet with the problem number.

3. Put your number, not your name, in the upper right hand corner of each page. If you have not received a number, please choose one (1234 for instance) and notify the graduate secretary as to which number you have chosen.

4. Each problem is worth the same number of points. There are six problems.

Analysis Prelim **August 2010**

1. Let H be a complex Hilbert space, with inner product $\langle \cdot, \cdot \rangle$. Let $S = \{\sigma_n \,|\, n \in \mathbb{Z}^+\}$ be an orthonormal *set* in H. Given $f \in H$, define *the nth partial sum* $S_N(f)$ *of* f *(with respect to* S*)* by

$$S_N(f) = \sum_{n=1}^{N} \langle f, \sigma_n \rangle \sigma_n.$$

(a) Show that, for any $f \in H$, $N \in \mathbb{Z}^+$, and complex numbers $d_1, d_2, \ldots, d_n$,

$$||f - S_N(f)|| \leq ||f - \sum_{n=1}^{N} d_n \sigma_n||.$$

Hint: first show, using properties of norms and inner products, that

$$||f - \sum_{n=1}^{N} d_n \sigma_n||^2 = ||f||^2 + \sum_{n=1}^{N} (|d_n - \langle f, \sigma_n \rangle|^2 - |\langle f, \sigma_n \rangle|^2).$$

(b) Use part (a) to show that

$$\lim_{N \to \infty} ||f - S_N(f)||$$

exists. Hint: compare $||f - S_N(f)||$ to $||f - S_{N-1}(f)||$.

(c) For H and S as above, suppose

$$\text{span}(S) := \{\text{finite } \mathbb{C}\text{-linear combinations of elements of } S\}$$

is dense in H. Show that

$$\lim_{N \to \infty} ||f - S_N(f)|| = 0$$

for all $f \in H$. Hints: first, take a sequence of elements g_N of span(S) that converge in norm to f. You may as well assume that g_N is a linear combination of $\sigma_1, \sigma_2, \ldots \sigma_{M_N}$, where $M_1 < M_2 < M_3 < \ldots$. (Why can you assume this?) Now consider the subsequence $||f - S_{M_N}(f)||$ of $||f - S_N(f)||$.

(d) True or false: under the hypotheses of part (c) above, S is, in fact, an orthonormal *basis* for H. Please explain.

2. Let

$$G(x) = e^{-\pi x^2}.$$

(a) Show that

$$\int_{\mathbb{R}} G(x)\, dx = 1.$$

Hint: denote the integral in question by I. Write I^2 as a double integral; change to polar coordinates.

(b) Show that, for any bounded, measurable function f, and for any real number x at which f is continuous,

$$\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \int_{\mathbb{R}} f(x - y) G\left(\frac{y}{\epsilon}\right) dy = f(x).$$

Hint: use part (a).

3.

(a) Let μ be a finite measure on $\mathbb{R}^d$ and let $S_t := \{x \in \mathbb{R}^d \mid |x| = t\},\ t > 0$. Consider the set $A := \{t > 0 \mid \mu(S_t) > 0\} \subset (0, \infty)$. Prove that A has zero Lebesgue measure.

(b) Suppose, instead of the finiteness of μ, we only assume that it is finite on compact sets of $\mathbb{R}^d$. Is the Lebesgue measure of A still zero in this case? Please explain.

4.

(a) Let $f : \mathbb{R} \to \mathbb{R}$ be a measurable function. Is either of the following statements stronger than the other?

   (i) f is continuous almost everywhere;

   (ii) f agrees with a continuous function almost everywhere.

   Please justify your answer.

(b) Suppose $f : \mathbb{R} \to \mathbb{R}$ is monotone. Is it necessarily true that:

   (a) f agrees almost everywhere with an everywhere differentiable function?

   (b) f is almost everywhere differentiable?

   Please justify your answers.

5. Let X and Y be compact Hausdorff spaces; let $C(X \times Y)$, $C(X)$, and $C(Y)$ denote the spaces of continuous functions on $X \times Y$, X, and Y respectively.
Let $f \in C(X \times Y)$. Given $\varepsilon > 0$, show that there exist $h_1, h_2, \dots, h_n \in C(X)$ and $g_1, g_2, \dots, g_n \in C(Y)$ such that

$$\sup_{(x,y)\in X\times Y} \left| f(x, y) - \sum_{i=1}^{n} h_i(x) g_i(y) \right| < \varepsilon.$$

6. For $\ell = 0, 1, 2, \dots$, define $C^\ell(S) = \{f \in C^\ell(\mathbb{R}) \mid f(x + n) = f(x)\, \forall n \in \mathbb{Z}\}$. It is easy (and you don't need) to check that, if $f \in C^\ell(S)$, then $f' \in C^{\ell-1}(S)$ for $\ell > 0$.
Given $f \in C^\ell(S)$ for $\ell > 0$, define the norm of f by:

$$||f||_{C^\ell(S)} = \sup_{x\in\mathbb{R}} |f(x)| + \sup_{x\in\mathbb{R}} |f'(x)|.$$

One then checks (but you don't have to) that $C^\ell(S)$ is a normed vector space.
Let $L^2(S)$ denote the usual Hilbert space $L^2([0, 1])$. Show that, if T is a bounded operator from $L^2(S)$ to $L^2(S)$ such that:

(i) The image of $L^2(S)$ under T is contained in $C^1(S) \subset L^2(S)$;

(ii) As a map from $L^2(S)$ to $C^1(S)$, T is continuous;

then $T : L^2(S) \to L^2(S)$ is a compact operator.