*RETURN THIS COVER SHEET WITH YOUR EXAM AND SOLUTIONS!*

# Algebra

# Ph.D. Preliminary Exam

**January 2018**

*INSTRUCTIONS*:

1. Answer each question on a separate page. Turn in a page for each problem even if you cannot do the problem.

2. Put your number, not your name, in the upper right hand corner of each page. If you have not received a number, please choose one (1234 for instance) and notify the graduate secretary as to which number you have chosen.

3. There are 6 problems, each worth the same number of points. Please do them all.

**Algebra Prelim** **January 2018**

1. Let $G$ be the symmetric group $S_5$ and let $P$ be a Sylow 5-subgroup of $G$.

(i) Show that the normalizer, $N_G(P)$, of $P$ in $G$ has order 20.

(ii) In the special case where $P$ contains the 5-cycle (12345), find a set of generators for $N_G(P)$.

2. Let $G$ be a group and let $Z(G)$ be the center of $G$. An automorphism $\alpha \in \mathrm{Aut}(G)$ of $G$ is said to be *central* if for all $x \in G$ we have $x^{-1}\alpha(x) \in Z(G)$. Show that the central automorphisms form a subgroup $N$ of $\mathrm{Aut}(G)$, and also that $N \trianglelefteq \mathrm{Aut}(G)$.

3. Let $k$ be a field and let $R$ be the subring of $k(x)$ generated by $k[x]$ and $1/x$. For a typical nonzero element $p(x) = \sum_{i=-M}^{N} a_i x^i$ of $R$, define

$$H(p(x)) = \max(\{i \in \mathbb{Z} : a_i \neq 0\}) \quad \text{and} \quad L(p(x)) = \min(\{i \in \mathbb{Z} : a_i \neq 0\}).$$

Show that $R$ is a Euclidean domain with Euclidean norm given by $N(p(x)) = H(p(x)) - L(p(x))$ (and $N(0) = 0$).

4. Let $F$ be a field of arbitrary characteristic. Show that any two elements of order 2 in the special linear group $SL_2(F)$ are conjugate in $GL_2(F)$. Find a necessary and sufficient condition on $F$ for $SL_2(F)$ to have a unique element of order 2.

5. Let $p$ be a prime, let $\mathbb{F}_p$ be the field with $p$ elements, and let $t$ be an indeterminate. Let $F = \mathbb{F}_p(t)$ be the field of fractions of the polynomial ring $\mathbb{F}_p[t]$.

(i) Show that $g(x) = x^p - x + t$ is separable over $F$.

(ii) Show that if $\alpha$ is a root of $g$, then $\alpha + 1$ is also a root. Deduce that the roots of $g$ are precisely those of the form $\alpha + b$ for $b \in \mathbb{F}_p$.

(iii) Show that $g$ has no roots in $F$.

(iv) Find the Galois group of $g$ over $F$.

6. Let $f(x)$ be a monic polynomial of degree $n > 0$ over a field $K$, and let $\Delta(f)$ denote its discriminant. Let $g(x) = f(x^2)$. You may assume without proof that $\Delta(g) = \Delta(f)^2(-4)^n f(0)$.

(i) Let $f(x) = x^2 + 3x + 1$, so that $g(x) = x^4 + 3x^2 + 1$. Show that $g$ is irreducible over $\mathbb{Q}$. [Hint: you may find it helpful to consider the roots of $f$ and of $g$.]

(ii) To which familiar group is the Galois group of $x^4 + 3x^2 + 1$ over $\mathbb{Q}$ isomorphic?