RETURN THIS COVER SHEET WITH YOUR EXAM AND SOLUTIONS!

Algebra
Ph.D. Preliminary Exam

January 2017

INSTRUCTIONS:

1. Answer each question on a separate page. Turn in a page for each problem even if you cannot do the problem.

2. Label each answer sheet with the problem number.

3. Put your number, not your name, in the upper right hand corner of each page. If you have not received a number, please choose one (1234 for instance) and notify the graduate secretary as to which number you have chosen.

4. There are 6 problems, each worth 17 points.

Algebra Prelim **January 2017**

1) Show that every group of size greater than 2 has a non-trivial automorphism (i.e., an automorphism other than the identity).

2) Suppose every maximal subgroup of a finite group G has prime index.

(a) Show that G has a normal Sylow subgroup. [Hint: Think about how a Sylow subgroup of the largest prime divisor sits inside a maximal subgroup.]

(b) Prove that G is solvable.

3) Let $\mathbb{R}[[x]]$ denote the ring of formal power series over x with real coefficients. That is, the elements in $\mathbb{R}[[x]]$ are of the form $\sum_{k=0}^{\infty} a_k x^k$ for $a_k \in \mathbb{R}$.

(a) Determine the units in $\mathbb{R}[[x]]$.

(b) Show that $\mathbb{R}[[x]]$ is a principal ideal domain.

4) Show that for every complex $n \times n$ matrix A,

$$\det(\exp(A)) = e^{\mathrm{trace}(A)}$$

where

$$\exp(A) = 1 + A + \frac{1}{2}A^2 + \cdots + \frac{1}{k!}A^k + \ldots$$

5) True/False. Either prove true or provide a counterexample.

(a) If $\mathbb{L}/\mathbb{K}$ and $\mathbb{K}/\mathbb{F}$ are algebraic, then $\mathbb{L}/\mathbb{F}$ is algebraic.

(b) If $\mathbb{L}/\mathbb{K}$ and $\mathbb{K}/\mathbb{F}$ are splitting fields, then $\mathbb{L}/\mathbb{F}$ is a splitting field.

6)

(a) Find an irreducible polynomial $f(x) \in \mathbb{Q}[x]$ whose splitting field over $\mathbb{Q}$ has a 12 element Galois group where all Sylow subgroups are normal.

(b) For your splitting field in (a) give the Sylow subgroups of the Galois group, up to isomorphism.