RETURN THIS COVER SHEET WITH YOUR EXAM AND SOLUTIONS!

Algebra
Ph.D. Preliminary Exam

January 2015

INSTRUCTIONS:

1. Answer each question on a separate page. Turn in a page for each problem even if you cannot do the problem.

2. Label each answer sheet with the problem number.

3. Put your number, not your name, in the upper right hand corner of each page. If you have not received a number, please choose one (1234 for instance) and notify the graduate secretary as to which number you have chosen.

4. There are 6 problems, each worth 17 points.

Algebra Prelim **January 2015**

1) True/False. Either show the following are true or supply a counter-example. Let G be a group.

(a) If H and K are subgroups, then either HK or KH is a subgroup.

(b) Every finite group G is isomorphic to a subgroup of $\mathrm{GL}_n(\mathbb{C})$ for some $n \in \mathbb{Z}_{\geq 1}$.

2) Let $G = H \ltimes U$ be a finite group, for groups H and U. Let p be prime, and let $\mathrm{Syl}_p(G)$ denote the set of Sylow p-subgroups of G.

(a) Show that if $\mathrm{Syl}_p(G) \cap \mathrm{Syl}_p(U) \neq \emptyset$, then $\mathrm{Syl}_p(G) = \mathrm{Syl}_p(U)$.

(b) Suppose $\mathrm{Syl}_p(G) \cap \mathrm{Syl}_p(U) \neq \emptyset$ and $\gcd(|H|, |U|) = 1$. Prove that H acts transitively on $\mathrm{Syl}_p(U)$ if and only if $Q \trianglelefteq U$ for some $Q \in \mathrm{Syl}_p(G)$.

3) Let R be the subring of $\mathbb{Q}$ consisting of fractions with odd denominators in reduced form; you may assume without proof that R is a ring.

(a) Prove that all irreducible elements of R, and all prime elements of R, are of the form $2u$ for some invertible element u of R.

(b) Prove that R is Euclidean. [Hint: consider the function $v : R\backslash\{0\} \to \mathbb{Z}$ for which $v(2^k u) = k$ whenever u is a unit.]

4) Let F be a field of characteristic zero; in this problem all matrices have entries in F. For a positive integer n, let I be the $n \times n$ identity matrix, and let J be the $n \times n$ matrix all of whose entries are equal to 1. Find the Jordan canonical form of $J - I$, and deduce that $J - I$ is invertible for all $n \geq 2$.

5) Let p be a prime number, $\mathbb{F}_p$ be the field with p elements, and $F = \mathbb{F}_p(t)$, where t is an indeterminate.

(a) Prove that the polynomial $x^p - t$ is irreducible over F. [Hint: consider factorizations over the polynomial ring $\mathbb{F}_p[t]$ and use Gauss's Lemma.]

(b) Let α be a root of $x^p - t$, and let $E = F(\alpha)$. Find the degree of E over F, and find the automorphism group $\mathrm{Aut}(E/F)$.

6) Suppose F is a field, K is the splitting field of a degree 4 separable polynomial in $F[x]$, and $[K : F] = 8$.

(a) Find $\mathrm{Gal}(K/F)$, up to isomorphism.

(b) How many degree two subextensions are there of K/F?