RETURN THIS COVER SHEET WITH YOUR EXAM AND SOLUTIONS!

Algebra

Ph.D. Preliminary Exam

January, 2014

INSTRUCTIONS:

1. Answer each question on a separate page. Turn in a page for each problem even if you cannot do the problem.

2. Label each answer sheet with the problem number.

3. Put your number, not your name, in the upper right hand corner of each page. If you have not received a number, please choose one (1234 for instance) and notify the graduate secretary as to which number you have chosen.

Algebra Prelim **January 2014**

1. Answer the following questions.

   (a) Let H be a subgroup of G and let X be the set of all left cosets of H in G. Show that there exists a normal subgroup N of G such that $N \subseteq H$ and G/N is isomorphic to a subgroup of the symmetric group on X.

   (b) Suppose that G is a group of order 10 and G has a subgroup of order 2 that is not normal. Show that G is isomorphic to a subgroup of S_5, the symmetric group on five elements.

2. Show that every group of order 143 is cyclic. [Give an appropriate full justification.]

3. Either prove or disprove the following statement, with a full justification.

   If R is an integral domain that is not a field, then the polynomial ring $R[x]$ can never be a principal ideal domain.

4. Consider the vector space $V \stackrel{\text{def}}{=} \mathbb{R}^2$ over $\mathbb{R}$ and the linear transformation $T : V \to V$ that projects onto the y-axis. Consider the polynomial ring $\mathbb{R}[t]$ in one variable t over $\mathbb{R}$ and form the $\mathbb{R}[t]$-module on V using T. That is, $fv = f(T)(v)$ for all $f \in \mathbb{R}[t]$ and all $v \in V$. Show that the only $\mathbb{R}[t]$-submodules are V, 0, the x-axis, and the y-axis.

5. Suppose that F is a subfield of an algebraically closed field K, and that -1 is not a sum of finitely many squares in F. (A square itself is also regarded as a sum of squares.)

   (a) Using Zorn's lemma, show that F has an algebraic extension G in which -1 is still not a sum of finitely many squares, while it is a sum of finitely many squares in any proper algebraic extension of G.

   (b) Taking G as in (a), show that every sum of finitely many squares in G is a square. [*Hint.* Suppose that $a \in G$ is a sum of finitely many squares of G, but is not a square. Adjoin a root of $x^2 - a$ to G.]

6. Answer the following questions.

   (a) Fix an integer $n \geq 1$. Let $I = \{1, 2, \ldots, n\}$ and let G be a subgroup of S_n, the symmetric group on I. Define an equivalence relation on I in the following way: for any $a, b \in I$,

   $$a \sim b \text{ iff } \; a = b \text{ or the transposition } (a\ b) \text{ lies in } G.$$

   First, show that this is an equivalence relation on I. Second, note that S_n and whence also G naturally act on I. In addition, if G acts transitively on I, then show that all the equivalence classes under $\sim$ have the same number of elements.

   (b) Let $f(x) \in \mathbb{Q}[x]$ be irreducible and have prime degree $p \geq 5$. Suppose that f has exactly $p-2$ real roots and 2 nonreal roots. Then find the Galois group of f over $\mathbb{Q}$. [*Hint.* Use (a) above.]