RETURN THIS COVER SHEET WITH YOUR EXAM AND SOLUTIONS!

Algebra

Ph.D. Preliminary Exam
Department of Mathematics
University of Colorado Boulder

January, 2013

INSTRUCTIONS:

1. Answer each of the six questions on a separate page. Turn in a page for each problem even if you cannot do the problem.

2. Label each answer sheet with the problem number.

3. Put your number, not your name, in the upper right hand corner of each page.

Algebra Preliminary Exam, January 2013, First Draft

1. Suppose $G = A \rtimes H$ is a finite group and A is abelian. Prove that the size of the conjugacy class of $a \in A$ in G is $|H : C_H(a)|$.

2. Let $H, K \lhd G$, where G is a finite group.

   (a) For each $P \in \mathrm{Syl}_p(HK)$, show that $P \cap H \in \mathrm{Syl}_p(H)$, $P \cap K \in \mathrm{Syl}_p(K)$, and $P = (P \cap H)(P \cap K)$.

   (b) Show that if H and K are nilpotent, then HK is nilpotent.

3. Prove that the subring of $\mathbb{Q}[x]$ consisting of all polynomials with integer constant term is not a UFD.

4. Does there exist a 6×6 matrix A

   (a) over $\mathbb{Q}$

   (b) over $\mathbb{R}$,

   such that $A^4 + I = A^2 - I$? Prove your claim.

5. How many roots does the polynomial $x^{2013} - 1$ have in the field $\mathbb{F}_{67}$ (note that $2013 = 3 \cdot 11 \cdot 61$)?

6. Let F be a field of characteristic 0. Show that if E/F is a normal field extension of prime degree p such that F contains the p^2th roots of unity, then E has an extension of degree p.