2012 January Algebra Preliminary Exam

1. True/False. Either show the following are true or supply a counter-example. Let G be a group with a subgroup H.

   (a) If P is a Sylow p-subgroup of G, then $P \cap H$ is a Sylow p-subgroup of H.

   (b) The number of Sylow p-subgroups of G is greater than or equal to the number of Sylow p-subgroups of H.

2. A group G is *Camina* if for every $g \in G \setminus [G, G]$, the coset $g[G, G]$ is a conjugacy class of G.

   (a) If G is a Camina group, is it the case that $[G, G] \subseteq Z(G)$ or $Z(G) \subseteq [G, G]$?

   (b) If G is a p-group with $[G, G] = Z(G)$ a cyclic group of prime order, then show that G is Camina.

3. Let R be a ring with unity $1 \neq 0$. A left R-module $V \neq \{0\}$ is called *irreducible* if its only left R-submodules are $\{0\}$ and V.

   (a) Show that if every nonzero element in R has a left inverse, then R is a division ring.

   (b) Show that R is a division ring if and only if the left R-module R is an irreducible R-module.

   (c) Show that R is a division ring if and only if every nonzero left R-module has a submodule isomorphic to the left R-module R.

4. Let p be a prime, and $G = GL_2(\mathbb{F}_p)$ (the invertible 2×2 matrices with entries in $\mathbb{F}_p$).

   (a) Writing the field $\mathbb{F}_{p^2}$ as a vector space over $\mathbb{F}_p$, explain how the multiplicative group of $\mathbb{F}_{p^2}$ is isomorphic to a subgroup of G. Show then that G has an element of order $p^2 - 1$.

   (b) Describe an element in G which is not conjugate in G to any matrix in Jordan canonical form.

5. Let p be a prime, and q a power of p. Let $\mathbb{F}_p$ and $\mathbb{F}_q$ denote respectively the finite fields with p and q elements. Let T denote the trace map from $\mathbb{F}_q$ to $\mathbb{F}_p$. (Recall that for a separable extension of fields L/K of degree n, if σ_i, $1 \leq i \leq n$, are the distinct embeddings of L into an algebraic closure $\bar{K}$ of K that fix K, then for any $a \in L$, $T(a) = \sum_{i=1}^{n} \sigma_i(a)$.)

   If $p = 2$, show that $T(x^2) = T(x)$ for all $x \in \mathbb{F}_q$, but if $p \neq 2$, then there is some $x \in \mathbb{F}_q$ such that $T(x^2) \neq T(x)$.

6. You are given that the polynomial
   $$f(x) = x^5 + 11x - 44$$
   is solvable by radicals over $\mathbb{Q}$, and has discriminant $2^{14}7^2 11^4$. Calculate the Galois group of f over the rationals. Show and justify all your work.